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SOME ANALYTICAL AND NUMERICAL RESULTS FOR CLADDED FUEL RODS SUB--ETC(U)
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FACULTY OF ENGINEERING AND APPLIED SCIENCES

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SOME ANALYTICAL AND NUMERICAL RESULTS FOR CLADDED FUEL RODS SUBJECTED TO THERMOIRRADIATION INDUCED CREEP

by

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ABSTRACT

Stresses and velocities are analyzed for a hollow cylinder of fuel encased in metallic cladding and subjected to high temperature and high neutron flux fields. The material is represented by a compressible nonlinear thermoirradiation viscoelastic model. A stress function for axisymmetric plane strain is introduced, and the problem essentially reduces to solving a nonlinear ordinary differential equation in the interfacial contact pressure. Some analytical results are obtained for the case of material properties independent of position. For the case of temperature, flux and thus material properties dependent on position, an approximate formulation is employed whereby the cylinder is divided into discrete rings with constant properties.

LIST OF FIGURES

NOTATION

a	radius of central hole in core, Fig. 1
$A_{II},B_{II},C_{II},D_{II},E_{II},$ G_{II},H_{II},I_{II}	coefficients in contact pressure equation for Case II, eq. (51.a)
$A_{III},B_{III},C_{III},D_{III},E_{III},$ G_{III},H_{III},I_{III}	coefficients in contact pressure equation for Case III, eq. (89.a)
b	outside radius of core, Fig. 1
B_I,C_I,E_I,H_I,I_I	coefficients in contact pressure equation for Case I, eq. (29.a)
$c_1(t),c_2(t),c_3(t),$ $c_4(r),c_5(r)$	integration functions, eqs. (26),(49)
$C_{Rs},C_{Rt},C_{Ts},C_{Tt}$	creep constants, eq. (5)
D	differential operator, $\partial/\partial t$
E	Young's modulus
f	interfacial contact pressure, eq. (13.b)
$F=\tilde{C}_{Rs}\Gamma_R+\tilde{C}_{Ts}\Gamma_T$	prescribed function of r, eq. (70)
F_R,F_T	functions of temperature and flux, eqs. (8)
F_{Ro},F_{To}	mean values of F_R, F_T
$g_1(t),g_2(t),g_4(r)$	integration functions, Equations (73-76)
h	thickness of cladding, Fig. 1
H	stress function
I_1	first stress invariant
J_2	second invariant of stress deviator
$J_{Rs},J_{Rt},J_{Ts},J_{Tt}$	creep compliance functions, eq. (5)
k	nondimensional constant, eq. (44)

$K_1, K_2, K_3, K_4, K_5, K_6$	nondimensional material constants, eqs. (38) and (63)
m	creep power in cladding
$m_{Rs}, m_{Rt}, m_{Ts}, m_{Tt}$	creep powers, eq. (5)
$n=2m+1$	creep power in one-dimensional test, eq. (11.a)
N	number of rings in Case III
$p(t)$	internal pressure, Fig. 1
p_o	constant internal pressure
P	differential operator, eqs. (16)
r	radial coordinate
R_i, S_i	functions for i-th ring, eqs. (83)
s_{ij}	stress deviator
t	time
T	temperature
u	radial displacement
v	radial velocity
X_1, X_2, X_3	functions of material constants for Case II, eqs. (49)
α_I, δ_I	coefficients of nondimensional contact pressure equation for Case I, eq. (33)
$\alpha_{II}, \beta_{II}, \gamma_{II}, \delta_{II}, \rho_{II}$	coefficients of nondimensional contact pressure equation for Case II, eq. (55)
$\alpha_{III}, \beta_{III}, \gamma_{III}, \delta_{III}, \rho_{III}$	coefficients of nondimensional contact pressure equation for Case III, eq. (92.a)
Γ_R, Γ_T	functions of r, eq. (23)
Δ	material function of r, eq. (71.b)
δ_{ij}	Kronecker delta

ε	strain
η_R, η_T	irradiation and thermal reduced times, eq. (5)
λ_R, λ_T	one-dimensional irradiation and thermal steady creep parameters, eq. (12)
μ_R, μ_T	one-dimensional irradiation and thermal transient creep parameters, eq. (12)
ν	Poisson's ratio
σ	stress
τ_R, τ_T	irradiation and thermal retardation times, eq. (5)
ϕ	flux
$\Phi_R(T), \Phi_T(T)$	functions of temperature, eqs. (7)
$\Psi_R(\phi), \Psi_T(\phi)$	functions of the flux, eqs. (7)
$\omega, \omega_1, \omega_2$	functions of material constants, eqs. (26) and (49.b)
$(\ )_I$	indicates quantity in Case I
$(\ )_{II}$	indicates quantity in Case II
$(\ )_{III}$	indicates quantity in Case III
$(\ )^c$	indicates quantity in cladding
$(\ )_R, (\ )_T$	indicates quantities in irradiation and thermal creep
$(\ )_s, (\ )_t$	indicate steady and transient creep quantities
$(\bar{\ })$	indicates nondimensionalized quantity
$(\tilde{\ })$	indicates modified material constants, eqs. (21),(27) and (46.c)

1. INTRODUCTION

A hollow cylinder of ceramic fuel (e.g., U_2O) encased in metallic cladding (e.g., stainless steel) is a widely used configuration for a nuclear fueld rod [1]. The cladding provides structural strength and also prevents the fuel from contaminating the coolent, and there is usually a gap between the fuel and the cladding at assembly. However, after the power of the reactor has been on for some time, this gap closes as a result of thermal expansion and swelling. An interfacial contact pressure is therefore developed between these two materials, and we may assume that displacements are continuous at the interface. The plenum pressure is transmitted to the central hole of the fuel, and this pressure may be time-dependent due to the immigration of gaseous products released from the nuclear fission reaction. The diameter of the central hole is assumed to be constant in the present study, since in practice its change is generally small. The geometry of the fuel rod is shown in Figure 1, where a is the radius of the central hole, b is the fuel radius, h is the cladding thickness, and $p(t)$ is the plenum pressure.

Creep material parameters are usually highly dependent on temperature. For problems with variable temperature fields Morland and Lee [2] employed the concept of a thermorheologically simple material, whereby a temperature dependent thermal reduced time ξ_T is introduced to convert the memory integral to one with constant material properties. Realizing the analogy between irradiation induced creep and thermally induced creep, Cozzarelli and Huang [3] introduced the concept of a fluxorheologically simple material, whereby a flux dependent irradiation

reduced time ξ_R is employed in the irradiation induced creep compliance function. Also, in dealing with the combined case of thermo-irradiation induced creep where both the temperature and flux fields are high, they used two reduced times where each is dependent on both temperature and flux. This approach results in two sets of memory integrals and differs from the approach employed by Rashid [4], where a single reduced time dependent on temperature and flux was used.

Experimental evidence [5] has shown that irradiation induced creep strain for both ceramic and metallic materials is essentially linearly dependent on stress. However, for thermal creep at moderate stress levels, although the strain for a ceramic material may still be linearly related to the stress, it is generally highly nonlinear for a metal. Therefore, we consider that the present problem consists of a linear viscoelastic cylinder encased in a nonlinear viscoelastic cylinder. Because the fuel rod is long, this can be treated as an axisymmetric plane strain problem. Also, since the cladding thickness is small in comparison with the radius of the fuel ($h \ll b$), membrane theory can be applied to the outer cylinder. For simplicity, we neglect the thermal expansion and swelling terms in the constitutive relation, and also ignore inhomogeneity and cracking in the fuel material. Finally, we note that we do not assume incompressibility since irradiation induced creep may not occur with conservation of volume, as pointed out by Gilbert and Straalsund [6] and Gittus [7]. We follow the approach of Courtine, Cozzarelli and Shaw [8] and Cozzarelli and Huang [3], whereby Poisson coefficients are introduced for each component of creep. For simplicity we set all of these coefficients equal, but not necessarily equal to $1/2$.

Stresses and velocities in viscoelastic cylinders have been extensively studied in connection with solid rocket fuel propellants, e.g., see [9-13]. These analytical studies generally dealt with linear incompressible viscoelastic materials containing a pressurized central hole and in some cases reinforced by a thin outer elastic ring. Although these studies serve as a useful guide to the analysis of the present problem, the analytical methods of solution employed are not directly applicable here because of the increased complexity of the present fuel rod problem due to thermal gradients, flux gradients, nonlinearity and compressibility. Several comprehensive computer codes have been developed recently for the stress analysis of fuel rods, employing approximate formulations and numerical solution technique such as finite difference [14] and finite element methods [15-17]. Although such methods are necessary for the general problem, the development of analytical results for various special cases is also important in that such results provide additional insight into the significance of the various input parameters. The development of "exact" formulations and analytical solutions is a major goal of the present paper, but we do resort to approximate formulations and numerical solutions when necessary.

In section 2, the basic equilibrium, compatibility and constitutive relations, along with the associated boundary and initial conditions, are given for the fuel rod problem. A general solution procedure is discussed, whereby a stress function is introduced and a contact condition at the fuel-cladding interface is used to obtain a nonlinear ordinary differential equation in the interfacial contact

pressure. Once this equation is solved, the stress and velocity fields follow directly. In section 3, this general procedure is used to obtain nondimensional solutions in three special cases: I - material properties independent of position and transient creep excluded, II - material properties independent of position but with transient creep included, and III - material properties dependent on position and transient creep excluded. In case I analytical solutions are obtained, in case II "exact" equations for equal retardation times are solved by numerical techniques, and in case III an approximate formulation is obtained for the incompressible case by dividing the cylinder into discrete rings with constant properties. The nondimensional stress and velocity distributions in the radial direction and their variation with time are then discussed for the three cases in section 4, with attention given to the effect of varying the various nondimensional material parameter ratios and the Poisson coefficient.

2. BASIC EQUATIONS OF THE PROBLEM

2.1 Field Equations

The fuel rod is assumed to be infinitely long and in a state of axially symmetric plane strain. Thus the equilibrium equation in polar coordinates is given by

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (1)$$

where σ_r and σ_θ are the radial and transverse stress components and r is the radial coordinate. Also, the compatibility equation is

$$\frac{\partial \varepsilon_r}{\partial r} + \frac{\varepsilon_r - \varepsilon_\theta}{r} = 0 \qquad (2)$$

where ε_r and ε_θ are the radial and transverse strain components. Finally, the strain-displacement relations are given as

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad (3)$$

$$\varepsilon_\theta = \frac{u}{r}$$

where u is the radial displacement.

As discussed in [3], a structural element under stress and subjected to high neutron flux and temperature fields will experience both thermal and irradiation induced creep. A cylindrical fuel rod in a

reactor is an element where both these effects can be very significant. The total strain will be expressed as a summation of linear elastic, steady thermal creep, steady irradiation induced creep, transient thermal creep and transient irradiation induced creep components. A mechanical spring-dashpot model of the nonlinear generalized Kelvin type can be used to represent such a summation of strain components and this is shown in Fig. 2. The form of the corresponding constitutive relation for either the fuel or the cladding is taken from [3], and is written in integral form as

$$
\begin{aligned}
\epsilon_{ij} &= \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu} I_1 \delta_{ij}\right) \\
&+ C_{Ts}\int_o^t J_{Ts}(\eta_T-\eta_T')\frac{\partial}{\partial t'}\left\{\left[J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right]^{m_{Ts}}\left(\sigma_{ij} - \frac{\nu}{1+\nu} I_1\delta_{ij}\right)\right\}dt' \\
&+ C_{Rs}\int_o^t J_{Rs}(\eta_R-\eta_T')\frac{\partial}{\partial t'}\left\{\left[J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right]^{m_{Rs}}\left(\sigma_{ij} - \frac{\nu}{1+\nu} I_1\delta_{ij}\right)\right\}dt' \\
&+ C_{Tt}\int_o^t J_{Tt}(\eta_T-\eta_T')\frac{\partial}{\partial t'}\left\{\left[J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right]^{m_{Tt}}\left(\sigma_{ij} - \frac{\nu}{1+\nu} I_1\delta_{ij}\right)\right\}dt' \\
&+ C_{Rt}\int_o^t J_{Rt}(\eta_R-\eta_R')\frac{\partial}{\partial t'}\left\{\left[J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right]^{m_{Rt}}\left(\sigma_{ij} - \frac{\nu}{1+\nu} I_1\delta_{ij}\right)\right\}dt'
\end{aligned}
\tag{5}
$$

In order to limit the complexity of the problem we have assumed in eq. (5) that the thermal expansion and irradiation swelling are negligibly small, and have set all Poisson coefficients in both the fuel and cladding equal to ν . In the above equation, $I_1 = \sigma_{kk}$ is the first invariant of stress and $J_2 = \frac{1}{2} s_{ik}s_{ik}$ is the second

invariant of the stress deviator $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$. Subscripts Ts, Tt, Rs and Rt designate the thermal steady, thermal transient, irradiation steady and irradiation transient creep quantities, respectively. The creep powers m_{Ts}, m_{Tt}, m_{Rs} and m_{Rt} are non-negative integers, and the creep parameters C_{Rs}, C_{Rt}, C_{Ts} and C_{Tt} are positive constants. The various creep compliance functions are defined as

$$J_{Ts}[\eta_T(t)] = \eta_T(t) \tag{6.a}$$

$$J_{Rs}[\eta_R(t)] = \eta_R(t) \tag{6.b}$$

$$J_{Tt}[\eta_T(t)] = 1 - \exp\left[\frac{-\eta_T(t)}{\tau_T}\right] \tag{6.c}$$

$$J_{Rt}[\eta_R(t)] = 1 - \exp\left[\frac{-\eta_R(t)}{\tau_R}\right] \tag{6.d}$$

where $\eta_R(t)$, $\eta_T(t)$ are the irradiation and thermal reduced time scales and τ_R , τ_T are the retardation time constants. Also, the reduced times are defined as

$$\eta_R(t) = \int_0^t \Phi_R(T)\Psi_R(\phi)\, dt' \tag{7.a}$$

$$\eta_T(t) = \int_0^t \Phi_T(T)\Psi_T(\phi)\, dt' \tag{7.b}$$

where $\Phi_R(T)$, $\Psi_R(\phi)$, $\Phi_T(T)$ and $\Psi_T(\phi)$ are some functions of temperature T and flux ϕ . In the present analysis we assume that T and ϕ have reached steady state, thus $T = T(r)$, $\phi = \phi(r)$ and eqs. (7.a)

and (7.b) simplify to

$$\eta_R(t) = F_R t \tag{8.a}$$

$$\eta_T(t) = F_T t \tag{8.b}$$

where $F_R = \Phi_R \Psi_R$ and $F_T = \Phi_T \Psi_T$. (8.c)

Although the irradiation induced creep strain will be taken as linearly dependent on the stress, the thermal creep strain will in general be nonlinear. The constitutive relations, eq. (5), can then be rewritten for the case of axisymmetric plane strain in the form

$$\epsilon_r = \left[\frac{1+\nu}{E} + \frac{C_{Ts}F_T}{D}\left(J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right)^{m_{Ts}} + \frac{C_{Rs}F_R}{D} + \frac{C_{Tt}F_T/\tau_T}{D+F_T/\tau_T}\left(J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right)^{m_{Tt}} + \frac{C_{Rt}F_R/\tau_R}{D+F_R/\tau_R}\right]\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right] \tag{9.a}$$

$$\epsilon_\theta = \left[\frac{1+\nu}{E} + \frac{C_{Ts}F_T}{D}\left(J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right)^{m_{Ts}} + \frac{C_{Rs}F_R}{D} + \frac{C_{Tt}F_T/\tau_T}{D+F_T/\tau_T}\left(J_2 + \frac{1-2\nu}{6(1+\nu)} I_1^2\right)^{m_{Tt}} + \frac{C_{Rt}F_R/\tau_R}{D+F_R/\tau_R}\right]\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right] \tag{9.b}$$

where D is the differential operator $\partial/\partial t$, and where $\sigma_z = \nu(\sigma_r+\sigma_\theta)$ has been eliminated from the right hand side. Furthermore, experimental evidence [5] indicates that, although thermal creep strain is nonlinear in stress for metals, it may be linear for ceramic materials. Accordingly, in subsequent work we will set

$$m_{Ts} = m_{Tt} = m \quad \text{for cladding material} \tag{10.a}$$

$$m_{Ts} = m_{Tt} = 0 \quad \text{for core material.} \tag{10.b}$$

Following [3], the axial and lateral strain for a one-dimensional creep test $\sigma = \sigma_o\, U(t)$ ($U(t)$ is unit step function) may now be written as

$$\varepsilon_x = \left\{\frac{\sigma_o}{E} + \left(\frac{\sigma_o}{\lambda_T}\right)^n F_T t + \left(\frac{\sigma_o}{\mu_T}\right)^n (1-e^{-F_T t/\tau_T})\right.$$

$$\left. + \left(\frac{\sigma_o}{\lambda_R}\right) F_R t + \left(\frac{\sigma_o}{\mu_R}\right)(1-e^{-F_R t/\tau_R})\right\} U(t) \tag{11.a}$$

$$\varepsilon_y = -\nu\varepsilon_x \tag{11.b}$$

where

$$n = 2m+1, \quad \lambda_T = \frac{2^{m/(2m+1)}(1+\nu)^{(m+1)/(2m+1)}}{C_{Ts}^{1/(2m+1)}}, \quad \lambda_R = \frac{1+\nu}{C_{Rs}}$$

$$\mu_T = \frac{2^{m/(2m+1)}(1+\nu)^{(m+1)/(2m+1)}}{C_{Tt}^{1/(2m+1)}}, \quad \mu_R = \frac{1+\nu}{C_{Rt}}. \tag{12.a-d}$$

It will be convenient in later work to use n, λ_T, λ_R, μ_T and μ_R instead of m, C_{Ts}, C_{Rs}, C_{Tt}, and C_{Rt}.

We consider the case of a known plenum pressure $p(t)$ in the central hole suddenly applied to $t = 0$. The boundary conditions in this case (see Fig. 1) are then written as

$$\sigma_r(a,t) = -p(t) \tag{13.a}$$

$$\sigma_r(b,t) = -f(t) \tag{13.b}$$

where $f(t)$ is the unknown contact pressure at the interface of the fuel and the cladding, and where $p(0^-) = f(0^-) = 0$. The cylinder is initially stress-free and responds in a purely elastic manner at $t = 0$, and thus the initial condition may be given either as

$$\sigma_r(r,0^-) = 0 \tag{14.a}$$

or as

$$\sigma_r(r,0^+) = \frac{a^2b^2}{b^2-a^2}[p(0^+)(\frac{1}{b^2} - \frac{1}{r^2})-f(0^+)(\frac{1}{a^2} - \frac{1}{r^2})] \tag{14.b}$$

where the latter is simply the radial stress of a corresponding elastic problem. When transient terms are included we will require additional initial conditions at $t = 0^+$; such conditions may be generated from the governing differential equation with homogeneous conditions at $t = 0^+$ and this will be discussed in subsequent sections.

2.2 General Method of Solution

For the present plane strain problem a stress function H can be introduced via the definitions

$$\sigma_r = \frac{1}{r}\frac{\partial H}{\partial r} \quad , \quad \sigma_\theta = \frac{\partial^2 H}{\partial r^2} \tag{15.a,b}$$

thereby ensuring that equilibrium equation (1) is identically satisfied. Combining eqs. (2),(9) and (15), a compatibility equation in terms of

H is obtained as

$$\frac{\partial}{\partial r}\left\{P\left[(1-\nu)\frac{\partial^2 H}{\partial r^2} - \frac{\nu}{r}\frac{\partial H}{\partial r}\right]\right\} + \frac{1}{r}P\left[\frac{\partial^2 H}{\partial r^2} - \frac{1}{r}\frac{\partial H}{\partial r}\right] = 0 \qquad (16.a)$$

where P is the differential operator

$$P = \frac{1+\nu}{E}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \frac{F_T}{\tau_T}\right)\left(\frac{\partial}{\partial t} + \frac{F_R}{\tau_R}\right) + (C_{Ts}F_T + C_{Rs}F_R)\left(\frac{\partial}{\partial t} + \frac{F_T}{\tau_T}\right)\left(\frac{\partial}{\partial t} + \frac{F_R}{\tau_R}\right)$$

$$+ C_{Tt}\frac{F_T}{\tau_T}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \frac{F_R}{\tau_R}\right) + C_{Rt}\frac{F_R}{\tau_R}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \frac{F_T}{\tau_T}\right) . \qquad (16.b)$$

Since F_R and F_T are functions of T and ϕ which in turn are functions of r, differential equation (16) is third order in r with r-dependent variable coefficients. The order in time is third in general, but reduces to second if $\tau_R = \tau_T$ and to first if $\tau_T, \tau_R \to \infty$ (i.e., no transient creep). The solution for H with boundary and initial conditions (13,14) may be obtained by various analytical and numerical procedures, and this will be discussed in detail in later sections. The stresses then follow symbolically from eqs. (15.a) and (15.b) as

$$\sigma_r = \sigma_r(f(t), r, t) \qquad (17.a)$$

$$\sigma_\theta = \sigma_\theta(f(t), r, t) \qquad (17.b)$$

where $f(t)$ is to be determined by consideration of the cladding.

The cladding is thin $(h << b)$ and may thus be treated as a membrane, yielding via eq. (13.b)

$$\sigma_r^c = -f(t) \tag{18.a}$$

$$\sigma_\theta^c = \frac{b}{h}\, f(t) \tag{18.b}$$

where superscript c designates a variable in the cladding. (A variable without a superscript will always relate to the fuel.) Now, for plain strain the invariant term in eqs. (9.a) and (9.b) simplifies to

$$J_2 + \frac{1-2\nu}{6(1+\nu)}\, I_1^2 = \frac{1}{2}\left[(1-\nu)(\sigma_r^2 + \sigma_\theta^2) - 2\nu\sigma_r\sigma_\theta\right] \tag{19}$$

Substituting eqs. (18.a) and (18.b) into eq. (19), we obtain for the cladding

$$J_2^c + \frac{1-2\nu}{6(1+\nu)}(I_1^c)^2 = \frac{f^2}{2}\left[1 + \left(\frac{b}{h}\right)^2 - \nu\left(1 - \frac{b}{h}\right)^2\right]. \tag{20}$$

Then, substituting this result into constitutive relation (9.b), we obtain the relation for the tangential strain in the cladding as

$$\varepsilon_\theta^c = \left\{\frac{1+\nu}{E^c} + \frac{\tilde{C}_{Rs}^c}{D} + \frac{\tilde{C}_{Ts}^c f^{2m}}{2^m D}\left[1 + \left(\frac{b}{h}\right)^2 - \nu\left(1 - \frac{b}{h}\right)^2\right]^m + \frac{C_{Rt}^c/\tilde{\tau}_R^c}{D + 1/\tilde{\tau}_R^c} \right.$$
$$\left. + \frac{C_{Tt}^c/\tilde{\tau}_T^c}{D + 1/\tilde{\tau}_T^c}\left(\frac{f^{2m}}{2^m}\right)\left[1 + \left(\frac{b}{h}\right)^2 - \nu\left(1 - \frac{b}{h}\right)^2\right]^m\right\}\left[(1-\nu)\frac{b}{h} + \nu\right] f\ , \tag{21.a}$$

with modified constants

$$\tilde{C}_{Rs}^c = C_{Rs}^c\, F_R^c \qquad \tilde{C}_{Ts}^c = C_{Ts}^c\, F_T^c$$

$$\tilde{\tau}_R^c = \tau_R^c/F_R^c \qquad \tilde{\tau}_T^c = \tau_T^c/F_T^c \tag{21.b-e}$$

where F_R^c and F_T^c are assumed independent of r since the cladding is modeled as a membrane. Finally, we assume that the fuel and the cladding press against each other tightly such that the radial displacement at the interface is continuous, and thus we obtain from eq. (4) the contact condition

$$\epsilon_\theta^c = \epsilon_\theta(b) \tag{22}$$

The formulation of the problem is now complete, since a substitution of eqs. (9.a),(9.b),(17.a) and (17.b) into eq. (18.a) yields a nonlinear differential equation in f. Once this equation is solved, stresses follow directly from eqs. (17.a,b) and the radial velocity may be obtained from eq. (4) in conjunction with eq. (9.b). This general procedure will be applied in the next section to the following three special cases of fuel rod problems: I - material properties independent of position with no transient creep components, II - material properties independent of position but with transient creep components, and III - material properties dependent on position with no transient creep components.

3. SOLUTIONS IN SOME SPECIAL CASES

As previously mentioned, the thermal and irradiation creep compliance functions are functions of temperature T and neutron flux ϕ, which in turn are generally dependent on the position r. In order to account for this dependence, we express the functions F_R and F_T [see eqs. (8)] in the form of M-th order radial power series expansions about their mean values, i.e.,

$$F_R(r) = F_{Ro}\Gamma_R(r) = F_{Ro} \sum_{i=0}^{M} k_{Ri}\, r^i \tag{23.a}$$

$$F_T(r) = F_{To}\Gamma_T(r) = F_{To} \sum_{i=0}^{M} k_{Ti}\, r^i \tag{23.b}$$

where F_{Ro} and F_{To} are the mean values of $F_R(r)$ and $F_T(r)$ in the fuel, which in turn require that the mean values of Γ_R and Γ_T equal unity. In the previously described Cases I and II $\Gamma_R = \Gamma_T = 1$ and thus $\Gamma_R = \Gamma_{Ro}$ and $F_T = F_{To}$, but in Case III F_R and F_T are not constant but are taken as functions of r in accordance with eqs. (23).

3.1 Case I - Material Properties Independent of Position - Transient Creep Terms Excluded

In this case $m_{Ts} = m_{Tt} = 0$ and τ_T, $\tau_R \to \infty$, and constitutive relations (9) simplify for the fuel to

$$\epsilon_r = \left(\frac{1+\nu}{E} + \frac{\tilde{C}_{Rs} + \tilde{C}_{Ts}}{D}\right)[(1-\nu)\sigma_r - \nu\sigma_\theta] \tag{24.a}$$

$$\epsilon_\theta = \left(\frac{1+\nu}{E} + \frac{\tilde{C}_{Rs} + \tilde{C}_{Ts}}{D}\right)[(1-\nu)\sigma_\theta - \nu\sigma_r] \tag{24.b}$$

where

$$\tilde{C}_{Ts} = C_{Ts} F_{To} \qquad \tilde{C}_{Rs} = C_{Rs} F_{Ro} \tag{24.c}$$

Accordingly, the compatibility equation in the stress function H [eq. (16)] reduces to

$$\left[\frac{(1+\nu)}{E}\frac{\partial}{\partial t} + (\tilde{C}_{Rs} + \tilde{C}_{Ts})\right]\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right)\right]\right\} = 0 \tag{25}$$

This equation may be solved by successive integration to yield

$$H = c_1(t)\frac{r^2}{4} + c_2(t)\ln r + c_3(t) + c_4(r)e^{-\omega t} \tag{26}$$

where $\omega = \dfrac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})E}{(1+\nu)}$ and $c_1(t)$, $c_2(t)$, $c_3(t)$ and $c_4(r)$ are functions of integration.

Using eq. (26) in definition (15.a) with a prime indicating d/dr we obtain for the radial stress

$$\sigma_r = \frac{c_1(t)}{2} + \frac{c_2(t)}{r^2} + \frac{c_4'(r)}{r}e^{-\omega t}$$

from which we may conclude that $c_1(0^-) = c_2(0^-) = c_4'(r) = 0$ in order that initial condition (14.a) be satisfied at every r. We may now enforce boundary condition (13) with $p(0^-) = f(0^-) = 0$

to evaluate integration functions $c_1(t)$ and $c_2(t)$, and the stresses are obtained with the use of eqs. (15) as

$$\sigma_r = \frac{a^2b^2}{b^2-a^2}\left[p(t)\left(\frac{1}{b^2} - \frac{1}{r^2}\right)-f(t)\left(\frac{1}{a^2} - \frac{1}{r^2}\right)\right] \tag{27.a}$$

$$\sigma_\theta = \frac{a^2b^2}{b^2-a^2}\left[p(t)\left(\frac{1}{b^2} + \frac{1}{r^2}\right)-f(t)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right] \tag{27.b}$$

Note that eqs. (27) are in the same form as the elastic solutions [e.g., see Eq. (14.b)], but here $f(t)$ is strongly dependent on the creep properties. The circumferential strain at $r = b$ in the fuel now follows from eqs. (24.b) and (27) as

$$\varepsilon_\theta(b) = \left[\frac{1+\nu}{E} + \frac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})}{D}\right]\left[\frac{(1-\nu)2a^2}{(b^2-a^2)}\, p(t) - \frac{(1-2\nu)b^2+a^2}{(b^2-a^2)}\, f(t)\right] \tag{28}$$

Equating eqs. (28) and (21), we finally obtain for the contact pressure the nonlinear ordinary differential equation

$$B_I\dot{f} + C_I f + E_I f^{2m+1} = H_I\dot{p} + I_I p \tag{29.a}$$

where a dot indicates d/dt and

$$B_I = \frac{(1-\nu)\,\frac{b}{h} + \nu}{E^c} + \frac{a^2 + b^2(1-2\nu)}{E(b^2-a^2)}$$

$$C_I = \tilde{C}^c_{Rs}\left[(1-\nu)\frac{b}{h} + \nu\right] + \frac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})[a^2 + b^2(1-2\nu)]}{(b^2 - a^2)}$$

$$E_I = \frac{\tilde{C}^c_{Ts}}{(1+\nu)2^m}\left[1 + \left(\frac{b}{h}\right)^2 - \nu\left(1 - \frac{b}{h}\right)^2\right]^m$$

$$H_I = \frac{2a^2(1-\nu)}{E(b^2-a^2)}$$

$$I_I = \frac{2a^2(\tilde{C}_{Rs} + \tilde{C}_{Ts})(1-\nu)}{(b^2-a^2)(1+\nu)} \tag{29.b-f}$$

with subscript I used to identify quantities in Case I.

The pressure in the central hole of the fuel is now assumed as constant after being suddenly applied to $t = 0$, i.e., $p(t) = p_o U(t)$. In order to obtain the solution to eq. (29.a) for $f(t)$, we require an initial value of $f(t)$. This can be obtained by integrating eq. (29) with respect to t from 0^- to 0^+ and setting $f(0^-) = 0$. Then, the problem is reformulated as

$$B_I \dot{f} + C_I f + E_I f^{2m+1} = I_I p_o, \quad t > 0 \tag{30}$$

with the initial condition

$$f(0^+) = \frac{H_I p_o}{B_I} = \frac{2a^2(1-\nu)E^c p_o}{[(1-\nu)(b/h)+\nu]E(b^2-a^2)+[a^2+b^2(1-2\nu)]E^c} \tag{31}$$

It is difficult to assign typical numerical values to the coefficients in eq. (30), since they contain creep constants which vary over very large ranges. However, if this equation is rewritten in nondimensional form we need only specify ratios of these creep constants, and the analysis of the solution is greatly facilitated. Thus we introduce the nondimensional variables

$$\bar{f} = \frac{f}{f(0^+)} \qquad \bar{t} = \frac{t\, E(\tilde{C}_{Rs} + \tilde{C}_{Ts})}{(1+\nu)} \tag{32.a,b}$$

and eqs. (30,31) become

$$\dot{\bar{f}} + \alpha_I \bar{f} + \delta_I \bar{f}^{2m+1} = 1 \, , \qquad \bar{t} > 0 \tag{33}$$

$$\bar{f}(0^+) = 1 \tag{34}$$

where the dot now indicates $d/d\bar{t}$. In eq. (33) the nondimensional coefficients are given by

$$\alpha_I = \frac{\bar{E}[(\bar{a}^2 + 1-2\nu)\bar{h} + K_1(1-\nu+\nu\bar{h})(1-\bar{a}^2)]}{(1-\nu + \nu\bar{h})(1-\bar{a}^2)+(\bar{a}^2+1-2\nu)\bar{E}\bar{h}} \tag{35}$$

$$\delta_I = \frac{2^{2m}K_2\bar{a}^{-4m}(1-\nu)^{2m}(1+\nu)^{m}\bar{E}^{2m+1}(1-\bar{a}^2)[1+\bar{h}^2-\nu(\bar{h}-1)^2]^m(1-\nu+\nu\bar{h})}{[(1-\nu+\nu\bar{h})(1-\bar{a}^2)+(\bar{a}^2+1-2\nu)\bar{E}\bar{h}]^{2m+1}} \tag{36}$$

where

$$\bar{a} = \frac{a}{b} \qquad \bar{h} = \frac{h}{b} \tag{37.a,b}$$

are nondimensional geometric constants; and

$$\bar{E} = \frac{E^c}{E} \tag{38.a}$$

$$K_1 = \frac{\dfrac{F^c_{Ro}}{\lambda^c_R}}{\dfrac{F_{Ro}}{\lambda_R} + \dfrac{F_{To}}{\lambda_T}} \tag{38.b}$$

$$K_2 = \frac{F^c_T\left(\dfrac{1}{\lambda^c_T}\right)^n p_o^{\,n-1}}{\dfrac{F_{Ro}}{\lambda_R} + \dfrac{F_{To}}{\lambda_T}} \tag{38.c}$$

are nondimensional material constants. Note that K_1 and K_2 have been expressed in terms of one-dimensional creep constants through the use of eqs. (12.a-d) in conjunction with definitions (21.a, 24.c).

The solution to differential equation (33) is easily obtained in implicit form as

$$\bar{t} = \int_1^{\bar{f}} \frac{dX}{1-\alpha_I X - \delta_I X^{2m+1}} , \qquad m = 0,1,2,\cdots \tag{39}$$

Since $\dot{\bar{f}} \to 0$ as $\bar{t} \to \infty$, an algebraic equation in the asymptotic contact pressure follows from eq. (33) as

$$\bar{f}(\infty)\left[\alpha_I + \delta_I \bar{f}(\infty)^{2m}\right] = 1 , \qquad m = 0,1,2,\cdots \tag{40}$$

In the special linear case $m = 0$ an explicit solution is obtained from eq. (33) as

$$\bar{f} = \frac{1-[1-(\alpha_I+\delta_I)]\ \exp[-(\alpha_I+\delta_I)\bar{t}]}{\alpha_I+\delta_I} , \qquad m = 0 \tag{41.a}$$

where we see that

$$\bar{f}(\infty) = \frac{1}{\alpha_I+\delta_I} , \qquad m = 0. \tag{41.b}$$

We may now express the nondimensional stress and velocity solutions in terms of $\bar{f}$. First we let

$$\bar{r} = \frac{r}{b}\,, \qquad \bar{\sigma}_r = \frac{\sigma_r}{f(0^+)}$$

$$\bar{\sigma}_\theta = \frac{\sigma_\theta}{f(0^+)} \qquad \bar{u} = \frac{uE}{bf(0^+)} \tag{42.a-d}$$

where $\bar{r}$, $\bar{\sigma}_r$, $\bar{\sigma}_o$ and $\bar{u}$ are the nondimensional radial coordinate, radial stress, tangential stress and radial displacement respectively, with $f(0^+)$ given in eq. (31). Eqs. (27) then yield

$$\bar{\sigma}_r = \frac{\bar{a}^2}{1-\bar{a}^2}\left[k(1-\frac{1}{\bar{r}^2})-\bar{f}(\frac{1}{\bar{a}^2}-\frac{1}{\bar{r}^2})\right] \tag{43.a}$$

$$\bar{\sigma}_\theta = \frac{\bar{a}^2}{1-\bar{a}^2}\left[k(1+\frac{1}{\bar{r}^2})-\bar{f}(\frac{1}{\bar{a}^2}+\frac{1}{\bar{r}^2})\right] \tag{43.b}$$

where we have introduced the nondimensional constant

$$k = \frac{(1-\nu+\nu\bar{h})(1-\bar{a}^2)+(\bar{a}^2+1-2\nu)\bar{E}\bar{h}}{2(1-\nu)\bar{a}^2\bar{E}\bar{h}} \tag{44}$$

The initial values of stress $\bar{\sigma}_r(\bar{r},0^+)$, $\bar{\sigma}_\theta(\bar{r},0^+)$ and the asymptotic values $\bar{\sigma}_r(\bar{r},\infty)$, $\bar{\sigma}_\theta(\bar{r},\infty)$ follow directly from eqs. (34),(40) and (43). Finally, eqs. (4),(24.b) and (23) give for the nondimensional velocity $\bar{v} = \partial\bar{u}/\partial t$

$$\bar{v} = \frac{\bar{a}^2(1+\nu)}{(1-\bar{a}^2)}\left\{k\left[(1-2\nu)\bar{r}+\frac{1}{\bar{r}}\right]-\left[1+(1-\alpha_I)\bar{f}-\delta_I\bar{f}^{2m+1}\right]\left[\frac{(1-2\nu)\bar{r}}{\bar{a}^2}+\frac{1}{\bar{r}}\right]\right\} \tag{45}$$

where the initial and asymptotic values $\bar{v}(\bar{r},0^+)$ and $\bar{v}(\bar{r},\infty)$ again follow from eqs. (34) and (40) respectively.

In the next subsection we investigate the effect of the transient creep terms.

3.2 Case II - Material Properties Independent of Position - Transient Creep Terms Included

We now include both thermally and irradiation induced transient creep in the problem. For simplicity we shall set all the retardation times equal, and hence constitutive relations (9) for the fuel simplify in this case to

$$\varepsilon_r = \left[\frac{1+\nu}{E} + \frac{\tilde{C}_{Rs} + \tilde{C}_{Ts}}{D} + \frac{1}{\tilde{\tau}}\,\frac{C_{Rt} + C_{Tt}}{D+1/\tilde{\tau}}\right][(1-\nu)\sigma_r - \nu\sigma_\theta] \tag{46.a}$$

$$\varepsilon_\theta = \left[\frac{1+\nu}{E} + \frac{\tilde{C}_{Rs} + \tilde{C}_{Ts}}{D} + \frac{1}{\tilde{\tau}}\,\frac{C_{Rt} + C_{Tt}}{D+1/\tilde{\tau}}\right][(1-\nu)\sigma_\theta - \nu\sigma_r] \tag{46.b}$$

where we have used definitions (24.c) along with

$$\tilde{\tau} = \tau_T/F_{To} = \tau_R/F_{Ro} \tag{46.c}$$

Also, compatibility equation (16) becomes

$$\left[\frac{(1+\nu)}{E}\,\frac{\partial^2}{\partial t^2} + (\tilde{C}_{Rs} + \tilde{C}_{Ts})\frac{\partial}{\partial t} + \frac{(C_{Rt} + C_{Tt})}{\tilde{\tau}}\left(\frac{\partial}{\partial t} + \frac{1}{\tilde{\tau}}\right)\right]\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)\right]\right\} = 0 \tag{47}$$

which is now second order in t, and thus we must supplement initial condition (14.a) with the second initial condition

$$\frac{\partial\sigma_r}{\partial t}(r,0^-) = 0 \tag{48}$$

By successive integration eq. (47) is easily solved as

$$H = c_1(t)\frac{r^2}{4} + c_2(t)\ell nr + c_3(t) + c_4(r)e^{-\omega_1 t} + c_5(r)e^{-\omega_2 t} \tag{49.a}$$

where $c_1(t)$, $c_2(t)$, $c_3(t)$, $c_4(r)$ and $c_5(r)$ are integration functions and

$$\omega_1, \omega_2 = \frac{-X_2 \pm \sqrt{X_2^2 - 4X_1X_3}}{2X_1}$$

$$X_1 = \frac{1+\nu}{E}$$

$$X_2 = (\tilde{C}_{Rs} + \tilde{C}_{Ts}) + \frac{1}{\tau}\left(\frac{1+\nu}{E} + C_{Rt} + C_{Tt}\right)$$

$$X_3 = \frac{\tilde{C}_{Rs} + \tilde{C}_{Ts}}{\tau} \quad . \tag{49.b-e}$$

After using initial conditions (14a,48) to show that $c_4(r) = c_5(r) = 0$ and enforcing boundary conditions (13) we obtain the same form for the stress solutions as in Case I, i.e., eqs. (27). However, since the interfacial pressure is obtained through the contact condition which in turn involves the material behavior, the $f(t)$ in eqs. (27) for the present case is clearly not the same as in Case 1.

The circumferential strain in the core material at the position $r = b$ now follows from eqs. (27,46.b) as

$$\epsilon_\theta(b) = \left[\frac{1+\nu}{E} + \frac{\tilde{C}_{Rs} + \tilde{C}_{Ts}}{D} + \frac{1}{\tau}\,\frac{C_{Tt} + C_{Rt}}{D+1/\tau}\right]\left[\frac{(1-\nu)2a^2}{(b^2-a^2)}p(t) - \frac{(1-2\nu)b^2+a^2}{(b^2-a^2)}f(t)\right] \tag{50}$$

By enforcing the contact condition $\epsilon_\theta^c = \epsilon_\theta(b)$ with retardation times

in the cladding equal to those in fuel (i.e., $\tilde{\tau} = \tilde{\tau}_T^c = \tilde{\tau}_R^c$), we obtain from eqs. (21,50) the second order nonlinear differential equation

$$A_{II}\ddot{f} + B_{II}\dot{f} + C_{II}f + D_{II}\dot{(f^{2m+1})} + E_{II}f^{2m+1} = G_{II}\ddot{p} + H_{II}\dot{p} + I_{II}p \quad (51.a)$$

where

$$A_{II} = \frac{(1+\nu)[(1-\nu)\frac{b}{h} + \nu]}{E^c} - \frac{(1+\nu)[2b^2\nu-(a^2+b^2)]}{E(b^2-a^2)}$$

$$B_{II} = \frac{(1+\nu)[(1-\nu)\frac{b}{h} + \nu]}{E^c\tilde{\tau}} + \tilde{C}^c_{Rs}[(1-\nu)\frac{b}{h} + \nu] + \tilde{C}^c_{Rt}[(1-\nu)\frac{b}{h} + \nu]$$

$$- \frac{(1+\nu)[2b^2\nu-(a^2+b^2)]}{E(b^2-a^2)\tilde{\tau}} - \frac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})[2b^2\nu-(a^2+b^2)]}{(b^2-a^2)}$$

$$- \frac{(C_{Rt} + C_{Tt})[2b^2\nu-(a^2+b^2)]}{\tilde{\tau}(b^2-a^2)}$$

$$C_{II} = \frac{\tilde{C}^c_{Rs}}{\tilde{\tau}}[(1-\nu)\frac{b}{h} + \nu] - \frac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})[2b^2\nu-(a^2+b^2)]}{\tilde{\tau}(b^2-a^2)}$$

$$D_{II} = \frac{1}{2^m}\left[1+(\frac{b}{h})^2-\nu(1-\frac{b}{h})^2\right]^m\left[(1-\nu)\frac{b}{h} + \nu\right]\left[\tilde{C}^c_{Ts} + \frac{C^c_{Tt}}{\tilde{\tau}}\right]$$

$$E_{II} = \frac{\tilde{C}^c_{Ts}}{2^m\tilde{\tau}}\left[1+(\frac{b}{h})^2-\nu(1-\frac{b}{h})^2\right]^m\left[(1-\nu)(\frac{b}{h}) + \nu\right]$$

$$G_{II} = \frac{(1+\nu)(1-\nu)2a^2}{E(b^2-a^2)}$$

$$H_{II} = \frac{(1+\nu)(1-\nu)2a^2}{E(b^2-a^2)\tilde{\tau}} + \frac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})(1-\nu)2a^2}{(b^2-a^2)} + \frac{(C_{Rt} + C_{Tt})(1-\nu)2a^2}{\tilde{\tau}(b^2-a^2)}$$

$$I_{II} = \frac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})(1-\nu)2a^2}{\tilde{\tau}(b^2-a^2)} \quad (51.b\text{-}i)$$

with subscript II indicating constants in Case II.

In order to find the solution for f, we require two initial conditions, which are readily obtained by integrating eq. (51) twice with respect to t from 0^- to 0^+ and then setting $f(0^-) = \dot{f}(0^-) = 0$. If the pressure at the center hole is again maintained constant at p_o for $t > 0$, we obtain the following differential equation and initial conditions:

$$A_{II}\ddot{f} + B_{II}\dot{f} + C_{II}f + D_{II}(\dot{f^{2m+1}}) + E_{II}f^{2m+1} = I_{II}p_o\,, \qquad t > 0 \tag{52}$$

$$f(0^+) = \frac{G_{II}p_o}{A_{II}} \tag{53}$$

$$\dot{f}(0^+) = \frac{H_{II}p_o}{A_{II}} - \frac{B_{II}G_{II}p_o}{A_{II}} - \frac{D_{II}}{A_{II}}\left(\frac{G_{II}p_o}{A_{II}}\right)^{2m+1} \tag{54}$$

Note that $G_{II}/A_{II} = H_I/B_I$ [see eqs. (51) and (29)], and thus the initial contact pressure given by eq. (53) is identical with the result given by eq. (31) in Case I. A closed-form solution to eq. (52) is not generally obtainable, but after nondimensionalization a numerical solution may be obtained.

We shall employ the same definitions for the nondimensional contact pressure and the nondimensional time as in Case I [eqs. (32.a,b)]. Eqs. (52-54) then become

$$\ddot{\bar{f}} + \beta_{II}\dot{\bar{f}} + \alpha_{II}\bar{f} + \gamma_{II}(\dot{\bar{f}^{2m+1}}) + \delta_{II}\bar{f}^{2m+1} = \rho_{II}\,, \tag{55}$$

$$\bar{f}(0^+) = 1 \tag{56}$$

$$\dot{\bar{f}}(0^+) = 1 - \beta_{II} - \gamma_{II} \tag{57}$$

where $\dot{\bar{f}} = d\bar{f}/d\bar{t}$, and

$$\beta_{II} = \frac{(1-\nu+\nu\bar{h})(1-\bar{a}^2)(K_1+K_3+K_5)+(\bar{a}^2+1-2\nu)\bar{E}\bar{h}(1+K_3+K_5)}{[(1-\nu+\nu\bar{h})(1-\bar{a}^2)+(\bar{a}^2+1-2\nu)\bar{E}\bar{h}]} \tag{58}$$

$$\alpha_{II} = \frac{\bar{E}K_3K_1(1-\nu+\nu\bar{h})(1-\bar{a}^2)+\bar{E}K_3(\bar{a}^2+1-2\nu)}{[(1-\nu+\nu\bar{h})(1-\bar{a}^2)+\bar{E}\bar{h}(\bar{a}^2+1-2\nu)]} \tag{59}$$

$$\gamma_{II} = \frac{2^{2m}(1-\nu)^{2m}\bar{a}^{-4m}\bar{E}^{2m+1}[1+\bar{h}^2-\nu(\bar{h}-1)^2]^m(1-\nu+\nu\bar{h})(1-\bar{a}^2)(K_2+K_6)(1+\nu)^m}{[(1-\nu+\nu\bar{h})(1-\bar{a}^2)+\bar{E}\bar{h}(\bar{a}^2+1-2\nu)]^{2m+1}} \tag{60}$$

$$\delta_{II} = \frac{2^{2m}(1-\nu)^{2m}\bar{a}^{-4m}\bar{E}^{2m+1}[1+\bar{h}^2-\nu(\bar{h}-1)^2]^m(1-\nu+\nu\bar{h})(1-\bar{a}^2)(1+\nu)^m K_2K_3}{[(1-\nu+\nu\bar{h})(1-\bar{a}^2)+\bar{E}\bar{h}(\bar{a}^2+1-2\nu)]^{2m+1}} \tag{61}$$

$$\rho_{II} = K_3 \ . \tag{62}$$

In definitions (58-62) we find the nondimensional constants $\bar{a}$, $\bar{h}$, $\bar{E}$, K_1, K_2 previously defined in eqs. (37,38) as well as the additional nondimensional material constants

$$K_3 = \frac{\frac{1}{E}}{\tilde{\tau}\left(\frac{F_{Ro}}{\lambda_R} + \frac{F_{To}}{\lambda_T}\right)} \tag{63.a}$$

$$K_4 = \frac{\frac{1}{\mu_R^c}}{\tilde{\tau}\left(\frac{F_{Ro}}{\lambda_R} + \frac{F_{To}}{\lambda_T}\right)} \tag{63.b}$$

$$K_5 = \frac{\frac{1}{\mu_R} + \frac{1}{\mu_T}}{\tilde{\tau}\left(\frac{F_{Ro}}{\lambda_R} + \frac{F_{To}}{\lambda_T}\right)} \tag{63.c}$$

$$K_6 = \frac{\left(\frac{1}{\mu_T^c}\right)^n p_o^{\,n-1}}{\hat{t}\left(\frac{F_{Ro}}{\lambda_R} + \frac{F_{To}}{\lambda_T}\right)} . \tag{63.d}$$

Note that Case II degenerates to Case I as $\hat{t} \to \infty$, i.e., for $K_3 = K_4 = K_5 = K_6 = 0$ eq. (55) reduces to eq. (33). A standard computer subroutine was used at the SUNYAB Computer Center to obtain numerical solutions to nonlinear differential equation (55). The routine uses a predictor-corrector method for nonstiff equations, and a variable-order method for stiff equations. The latter method is used in dealing with our problem when m is large. The asymptotic value $\bar{f}(\infty)$ may be obtained analytically by setting $\ddot{\bar{f}} = \dot{\bar{f}} = 0$ in eq. (55), whereupon we get $\alpha_{II}\bar{f}(\infty) + \delta_{II}\bar{f}(\infty)^{2m+1} = \rho_{II}$ which can be shown to be identical with eq. (40) of Case I.

We now introduce the same definitions for the nondimensional variables $\bar{r}$, $\bar{\sigma}_r$, $\bar{\sigma}_\theta$ and $\bar{u}$ as in Case I [see eqs. (42.a-d)], with the additional definition for the nondimensional strain

$$\bar{\varepsilon}_\theta = \frac{E\varepsilon_\theta}{f(0^+)} \tag{64}$$

Using these definitions we of course obtain from eqs. (27) the same form for the nondimensional stress relations as in Case I [i.e., eqs. (43)]. The governing differential equation for the nondimensional circumferential strain is now obtained by substituting definition (64) and stress solutions (43) into eq. (46.b), and thus we obtain

$$\frac{\partial^2\bar{\varepsilon}_\theta}{\partial\bar{t}^2} + K_3\frac{\partial\bar{\varepsilon}_\theta}{\partial\bar{t}} = \frac{\bar{a}^2(1+\nu)}{(1-\bar{a}^2)}\left\{\left(\frac{1-2\nu}{\bar{a}^2} + \frac{1}{\bar{r}^2}\right)[-\ddot{\bar{f}}-\dot{\bar{f}}(1+K_3+K_5)-K_3\bar{f}]\right.$$

$$\left. + K_3(1-2\nu+\frac{1}{\bar{r}^2})k\right\} = G(\bar{r},\bar{t}) \tag{65}$$

where $G(\bar{r},\bar{t})$ is known in terms of the numerical solution for $\bar{f}(\bar{t})$. The initial conditions for $\bar{\varepsilon}_\theta$ are obtained by following the same procedure used in obtaining $\dot{f}(0^+)$ and $f(0^+)$ for eq. (52), and accordingly we obtain

$$\bar{\varepsilon}_\theta(\bar{r},0^+) = \frac{(1+\nu)\bar{a}^2}{(1-\bar{a}^2)}\left[k(1-2\nu+\frac{1}{\bar{r}^2})-(\frac{1-2\nu}{\bar{a}^2} + \frac{1}{\bar{r}^2})\right] \tag{66}$$

$$\frac{\partial}{\partial\bar{t}}\,\bar{\varepsilon}_\theta(\bar{r},0^+)= \frac{(1+\nu)\bar{a}^2}{(1-\bar{a}^2)}\left[(\beta_{II} + \gamma_{II}-2-K_5)(\frac{1-2\nu}{\bar{a}^2} + \frac{1}{\bar{r}^2})\right.$$

$$\left.+k(1+K_5)(1-2\nu + \frac{1}{\bar{r}^2})\right] \tag{67}$$

Finally, eq. (65) coupled with condition (67) can be solved by the method of integrating factor to obtain $\partial\bar{\varepsilon}_\theta/\partial\bar{t}$. By the use of eq. (4), we then find the nondimensional velocity as

$$\bar{v} = \frac{\partial\bar{u}}{\partial\bar{t}} = \bar{r}\,\frac{\partial\bar{\varepsilon}_\theta}{\partial\bar{t}} = \bar{r}\left[\frac{\partial}{\partial\bar{t}}\,\bar{\varepsilon}_\theta(\bar{r},0^+)e^{-K_3\bar{t}} + \int_{0^+}^{\bar{t}} e^{-K_3(\bar{t}-t')}G(\bar{r},t')dt'\right] \tag{68}$$

Note that the initial value of $\bar{v}(\bar{r},0^+)$ obtained with the use of eq. (67) is not the same as the result obtained by setting $\bar{t} = 0$ in eq. (45) of Case I. However, one can show that the asymptotic value $\bar{v}(\bar{r},\infty)$ [obtained from eq. (65) by setting $\partial^2\bar{\varepsilon}_\theta/\partial\bar{t}^2 = 0$, $\partial\bar{\varepsilon}_\theta/\partial\bar{t} = \bar{v}(\bar{r},\infty)/\bar{r}$, $\ddot{\bar{f}} = \dot{\bar{f}} = 0$ and $\bar{f} = \bar{f}(\infty)$] is identical with the result obtained in Case I by letting $\bar{t} \to \infty$ in eq. (45).

All solutions obtained in this section reduce to the corresponding solutions for Case I, when we let $\tilde{t} \to \infty$ and set $K_3 = K_4 = K_5 = K_6 = 0$. In the following subsection we again ignore transient creep, but introduce the effect of position dependent material properties.

3.3 Case III - Material Properties Dependent on Position - Transient Creep Terms Excluded

In this case, the material properties are not constant but are taken as dependent on the position in accordance with power series (23). The constitutive relations for the fuel are thus obtained from eq. (9) as

$$\varepsilon_r = \left[\frac{1+\nu}{E} + \frac{F(r)}{D}\right]\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right] \tag{69.a}$$

$$\varepsilon_\theta = \left[\frac{1+\nu}{E} + \frac{F(r)}{D}\right]\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right] \tag{69.b}$$

where using eq. (24.c)

$$F(r) = \tilde{C}_{Rs}\Gamma_R(r) + \tilde{C}_{Ts}\Gamma_T(r) \tag{70}$$

which is a prescribed function of r. In the present case, compatibility condition (16) becomes

$$\frac{\partial}{\partial r}\left\{\left[\frac{\partial}{\partial t} + \Delta(r)\right]\left[(1-\nu)\frac{\partial^2 H}{\partial r^2} - \frac{\nu}{r}\frac{\partial H}{\partial r}\right]\right\}$$

$$+ \frac{1}{r}\left\{\left[\frac{\partial}{\partial t} + \Delta(r)\right]\left[\frac{\partial^2 H}{\partial r^2} - \frac{1}{r}\frac{\partial H}{\partial r}\right]\right\} = 0 \tag{71.a}$$

where

$$\Delta(r) = \frac{EF(r)}{1+\nu} \tag{71.b}$$

The general solution to eq. (71.a) is not easily obtained, but in the special case of incompressible materials ($\nu=1/2$) it simplifies to the factored form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\left(\frac{\partial}{\partial t} + \Delta(r)\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial H}{\partial r}\right)\right] = 0 \tag{72}$$

Eq. (72) is now readily solved through the use of an integrating factor and successive integration, and thus we confine our attention to this special case in the remainder of this subsection. Accordingly, the stresses as defined in eqs. (15) are obtained as

$$\sigma_r = \int_a^r \frac{1}{r'^3}\int_{0+}^t g_2(t')e^{-\Delta(r')(t-t')}dt'dr' + \int_a^r g_4(r')e^{-\Delta(r')t}dr' + g_1(t) \tag{73.a}$$

$$\sigma_\theta = \int_a^r \frac{1}{r'^3}\int_{0+}^t g_2(t')e^{-\Delta(r')(t-t')}dt'dr' + \int_a^r g_4(r')e^{-\Delta(r')t}dr' + g_1(t) + \frac{1}{r^2}\int_{0+}^t g_2(t')e^{-\Delta(r')(t-t')}dt' + g_4(r)e^{-\Delta(r)t} \tag{73.b}$$

where $g_1(t)$, $g_2(t)$ and $g_4(r)$ are three integration functions. Expressions for these integration functions may be obtained by applying

conditions (13) and (14.b) to eq. (73.a), and accordingly

$$g_1(t) = -p(t) \tag{74}$$

$$g_4(r) = \frac{a^2b^2}{b^2-a^2}[p(0^+)-f(0^+)](\frac{2}{r^3}) \tag{75}$$

$$\int_a^b \frac{1}{r'^3} \int_{0^+}^t g_2(t')e^{-\Delta(r')(t-t')} dt'dr' = p(t)-f(t)-\frac{a^2b^2}{b^2-a^2}[p(0^+)-f(0^+)] \cdot \int_a^b \frac{2e^{-\Delta(r')t}}{r'^3} dr' \tag{76}$$

We see that although $g_1(t)$ and $g_4(r)$ are now determined via eqs. (74-75), eq. (76) must first be combined with the contact condition before $g_2(t)$ and $f(t)$ can be determined. The asymptotic values $\sigma_r(r,\infty)$ and $\sigma_\theta(r,\infty)$ are easily determined by setting $\partial/\partial t = 0$ in eq. (72) and repeating the above solution procedure, yielding

$$\sigma_r(r,\infty) = \int_a^r \frac{g_2(\infty)}{r'^3\Delta(r')} dr' + g_1(\infty) \tag{77.a}$$

$$\sigma_\theta(r,\infty) = \sigma_r(r,\infty) + \frac{g_2(\infty)}{r^2\Delta(r)} \tag{77.b}$$

In the present case the transient creep terms are absent and the Poisson coefficients are all equal to $1/2$; the equation for the circumferential strain in the cladding then follows from eq. (21) as

$$\varepsilon_\theta^c = \left[\frac{3}{2E^c} + \frac{\tilde{C}^c_{Rs}}{D} + \frac{\tilde{C}^c_{Ts}}{2^{2m}D}(1 + \frac{b}{h})^{2m}f^{2m}\right](1+ \frac{b}{h})\frac{f}{2} \tag{78}$$

Also, the circumferential strain in the fuel at $r = b$ is obtained from eq. (69.b), which becomes for $\nu = 1/2$

$$\varepsilon_\theta(r) = \left[\frac{3}{2E} + \frac{F(r)}{D}\right]\left(\frac{\sigma_\theta - \sigma_r}{2}\right) \qquad (79)$$

Then by substituting eqs. (73) for the stresses into eq. (79), we get with the use of definition (71.b) the simple result

$$\varepsilon_\theta(r) = \frac{3}{4Er^2 D}\, g_2(t) \qquad (80)$$

where we see that the assumption of incompressibility leads to a strain-rate which is separable in r and t. By equating eqs. (78) and (80) at $r = b$ (in accordance with contact condition (22)) we are able to obtain the expression

$$g_2(t) = \frac{4Eb^2}{3}\left[\frac{3}{2E^c}\left(1+\frac{b}{h}\right)\frac{\dot{f}}{2} + \tilde{C}^c_{Rs}\left(1+\frac{b}{h}\right)\frac{f}{2} + \frac{\tilde{C}^c_{Ts}}{2^{2m+1}}\left(1+\frac{b}{h}\right)^{2m} f^{2m+1}\right] \qquad (81)$$

which may now be combined with eq. (76).

Because the integrand on the left hand side of eq. (76) is not separable in r and t, repeated differentiation with respect to t will not yield a simple differential relation between $g_2(t)$ and $f(t)$. In order to obtain such a differential relation which in turn would yield a differential equation for the contact pressure when combined with eq. (81), we employ an approximate formulation in which the fuel core is divided into N concentric rings. In each ring the material properties are assumed to be constant, and we set

$$\Delta(r) = \Delta_i, \quad r_{i-1} \le r \le r_i \quad (i = 1,\cdots,N;\ r_o = a, r_N = b) \qquad (82)$$

where Δ_i is the mean value in the i-th ring. Using eq. (82) we

may approximate eq. (76) as

$$p(t) - f(t) = \sum_{i=1}^{N} S_i(t) \tag{83.a}$$

where

$$S_i(t) = \int_{r_{i-1}}^{r_i} \frac{1}{r'^3} \int_{0^+}^{t} g_2(t')e^{-\Delta_i(t-t')} dt'dr' + 2e^{-\Delta_i t} R_i \frac{a^2b^2}{(b^2-a^2)} \cdot [p(0^+)-f(0^+)]$$

$$R_i = \int_{r_{i-1}}^{r_i} \frac{dr'}{r'^3} = -\frac{1}{2}\left(\frac{1}{r_i^2} - \frac{1}{r_{i-1}^2}\right) \tag{83.b-c}$$

The integrals in the functions S_i are now separated in r and t, and thus the K-th order time derivatives of eq. (83.a) with $1 \leq K \leq N$ assume the simple form

$$p^{(K)} - f^{(K)} = (-1)^K \sum_{i=1}^{N} \Delta_i^K S_i + \sum_{j=1}^{K} (-1)^{j-1} g_2^{(K-j)}(t) \left[\sum_{i=1}^{N} \Delta_i^{j-1} R_i\right] \quad K=1,\cdots,N \tag{84}$$

where superscript (K) indicates the K-th time derivative of a variable.

Eqs. (83.a) and (84) constitute a set of N+1 equations in the N functions $S_i(t)$ and the integration function $g_2(t)$. This set may be reduced by simple algebraic elimination to a single differential relation between $g_2(t)$ and $f(t)$, with all the $S_i(t)$ eliminated. As a simple illustration we consider a two-ring approximation (i.e., N=2) with rings of equal thickness; the three equations given by eqs. (83.a) and (84)are now

$$p - f = \sum_{i=1}^{2} S_i \tag{85}$$

$$\dot{p} - \dot{f} = - \sum_{i=1}^{2} \Delta_i S_i + g_2(t) \sum_{i=1}^{2} R_i \tag{86}$$

$$\ddot{p} - \ddot{f} = \sum_{i=1}^{2} \Delta_i^2 S_i - g_2(t) \sum_{i=1}^{2} \Delta_i R_i + \dot{g}_2(t) \sum_{i=1}^{2} R_i \tag{87}$$

After eliminating S_1 and S_2 from the above equations, we obtain the following differential relation between g_2 and f :

$$(\ddot{p}-\ddot{f})+(\Delta_1+\Delta_2)(\dot{p}-\dot{f})+\Delta_1\Delta_2(p-f)=(R_1+R_2)\dot{g}_2(t)+(\Delta_1 R_2+\Delta_2 R_1)g_2(t) \tag{88}$$

This relation may now be combined with contact condition (81) to obtain the governing differential equation in the contact pressure for the two-ring approximation. Accordingly, we obtain

$$A_{III}\ddot{f} + B_{III}\dot{f} + C_{III}f + D_{III}(\dot{f^{2m+1}}) + E_{III}f^{2m+1} =$$

$$G_{III}\ddot{p} + H_{III}\dot{p} + I_{III}p \tag{89.a}$$

where

$$A_{III} = (R_1 + R_2)b^2 \frac{E}{E^c} (1+ \frac{b}{h}) + 1$$

$$B_{III} = (\Delta_1 R_2+\Delta_2 R_1)b^2 \frac{E}{E^c} (1+ \frac{b}{h})+(R_1+R_2)b^2 \frac{\tilde{C}^c_{Rs} 2E}{3}(1+ \frac{b}{h})+(\Delta_1+\Delta_2)$$

$$C_{III} = (\Delta_1 R_2+\Delta_2 R_1)b^2 \frac{\tilde{C}^c_{Rs} 2E}{3}(1+ \frac{b}{h})$$

$$D_{III} = (R_1+R_2)b^2 \frac{\tilde{C}^c_{Ts}E}{3\cdot 2^{2m-1}}(1+\frac{b}{h})$$

$$E_{III} = (\Delta_1 R_2+\Delta_2 R_1)b^2 \frac{\tilde{C}^c_{Ts}E}{3\cdot 2^{2m-1}}(1+\frac{b}{h})^{2m+1}$$

$$G_{III} = 1$$

$$H_{III} = (\Delta_1+\Delta_2)$$

$$I_{III} = \Delta_1 \Delta_2 \qquad (89.b\text{-}i)$$

with subscript III indicating constants in Case III.

Since eq. (89.a) is second order we require two initial conditions, which are obtained by the same procedure discussed for the previous cases as

$$f(0^+) = \frac{G_{III}p_o}{A_{III}} \qquad (90.a)$$

$$\dot{f}(0^+) = \frac{H_{III}p_o}{A_{III}} - \frac{B_{III}G_{III}p_o}{A_{III}} - \frac{D_{III}}{A_{III}}\left(\frac{G_{III}p_o}{A_{III}}\right)^{2m+1} \qquad (90.b)$$

For additional simplicity we shall also assume that the functions $\Gamma_R(r)$ and $\Gamma_T(r)$ in eq. (70) are equal in each ring of the two-ring approximation. Then in eqs. (89.b-i) the mean value of $\Delta(r)$ in the i-th ring becomes upon setting $\nu = 1/2$ in eq. (71.b)

$$\Delta_i = \Gamma_i \frac{(\tilde{C}_{Rs} + \tilde{C}_{Ts})2E}{3} \qquad i=1,2 \qquad (91)$$

where Γ_i is the corresponding mean value of $\Gamma_R(r)$ (or $\Gamma_T(r)$) in the i-th ring.

For the load we assume again that a constant pressure p_o is maintained at the central hole for $t>0$. After introducing the same nondimensional contact pressure $\bar{f}$ and time $\bar{t}$ as in Case I [eqs. (32)], eqs. (89.a) and (90) became

$$\ddot{\bar{f}} + \beta_{III}\dot{\bar{f}} + \alpha_{III}\bar{f} + \gamma_{III}\dot{\bar{f}}^{2m+1} + \delta_{III}\bar{f}^{2m+1} = \rho_{III} \tag{92.a}$$

$$\bar{f}(0^+) = 1 \tag{92.b}$$

$$\bar{f}(0^+) = 1 - \beta_{III} - \gamma_{III} \tag{92.c}$$

In the above $(\dot{\ }) = d/d\bar{t}$, and

$$\beta_{III} = \frac{\left[\frac{\Gamma_1(3+\bar{a})(1-\bar{a})}{(1+\bar{a})^2} + \frac{\Gamma_2(3\bar{a}+1)(1-\bar{a})}{(1+\bar{a})^2\,\bar{a}^2}\right](1+\bar{h}) + \frac{(1-\bar{a}^2)}{\bar{a}^2}(1+\bar{h})K_1\bar{E} + 2(\Gamma_1+\Gamma_2)\bar{E}\bar{h}}{\left[\frac{(1-\bar{a}^2)}{\bar{a}^2}(1+\bar{h}) + 2\bar{E}\bar{h}\right]}$$

$$\alpha_{III} = \frac{\bar{E}\left[\frac{\Gamma_1(\bar{a}+3)(1-\bar{a})}{(1+\bar{a})^2} + \frac{\Gamma_2(3\bar{a}+1)(1-\bar{a})}{(1+\bar{a})^2\bar{a}^2}\right]K_1(1+\bar{h}) + 2\Gamma_1\Gamma_2\bar{E}\bar{h}}{\left[\frac{(1-\bar{a}^2)}{\bar{a}^2}(1+\bar{h}) + 2\bar{E}\bar{h}\right]}$$

$$\gamma_{III} = \frac{\left(\frac{1-\bar{a}^2}{\bar{a}^2}\right)(1+\bar{h})^{2m+1}\;\bar{E}^{2m+1}\;3^m\;K_2}{\left[\left(\frac{1-\bar{a}^2}{\bar{a}^2}\right)(1+\bar{h}) + 2\bar{E}\bar{h}\right]^{2m+1}}$$

$$\delta_{III} = \frac{\left[\frac{\Gamma_1(3+\bar{a})(1-\bar{a})}{(1+\bar{a})^2} + \frac{\Gamma_2(3\bar{a}+1)(1-\bar{a})}{(1+\bar{a})^2\bar{a}^2}\right](1+\bar{h})^{2m+1}\;\bar{E}^{2m+1}\;3^m\;K_2}{\left[\left(\frac{1-\bar{a}^2}{\bar{a}^2}\right)(1+\bar{h}) + 2\bar{E}\bar{h}\right]^{2m+1}}$$

$$\rho_{III} = \Gamma_1\,\Gamma_2 \; . \tag{93.a-e}$$

For $\Gamma_1 = \Gamma_2 = 1$ second order differential equation (92.a) is equivalent to first order equation (33) of Case I (with $\nu = 1/2$) , since it may be shown that then eq. (92.a) is a linear combination of eq. (33) and its first derivative. We also see that eq. (92.a) is very similar in form to nonlinear second order equation (55) of Case II, and thus the same numerical procedure may be used in obtaining $\bar{f}$. Had we taken more than two rings (i.e., $N > 2$) in our approximate formulation, we would have obtained an N-th order differential equation in $\bar{f}$. However, the same numerical method can also be applied to the solution of this higher order equation. Additional initial conditions must of course be generated in accordance with the procedure previously described. As in the previous cases the asymptotic value $\bar{f}(\infty)$ may be obtained analytically by setting $\ddot{\bar{f}} = \dot{\bar{f}} = 0$ in eq. (92.a), yielding $\alpha_{III}\bar{f}(\infty) + \delta_{III}\bar{f}^{-2m+1}(\infty) = \rho_{III}$ which may then be used in eqs. (77). Note that since the present material is not equivalent to one with constant properties as $\bar{t} \to \infty$, the asymptotic value will in general be different from those in the previous cases.

Having obtained approximate values for the contact pressure and the integrating function $g_2(t)$ via eq. (81), the stresses are then obtained from eqs. (73-75). The best procedure would be to drop the multi-ring approximation at this point, and evaluate the integrals in eqs. (73) numerically over r using the actual distribution for $\Delta(r)$ and thereby obtain continuous distributions for both $\sigma_r(r)$ and $\sigma_\theta(r)$. However, a simpler estimate to σ_r may be obtained by also employing the step approximation to $\Delta(r)$ in eq. (73.a). Thus we nondimensionalize in accordance with eqs. (42) and employ the two-ring approximations to $g_2(t)$ and $\Delta(r)$ to obtain the estimate

$$\bar{\sigma}_r \approx \begin{cases} \frac{1}{2}\left(\frac{1}{\bar{a}^2} - \frac{1}{\bar{r}^2}\right) \int_{0+}^{\bar{t}} \left\{ \frac{(1+\bar{h})}{\bar{E}\bar{h}} \frac{d\bar{f}}{d\bar{t}'} + \frac{K_2 2^m (1+\bar{h})^{2m+1} \bar{E}^{2m}}{\bar{h}\left[\left(\frac{1-\bar{a}^2}{\bar{a}^2}\right)(1+\bar{h})+2\bar{E}\bar{h}\right]^{2m}} \bar{f}^{2m+1} \right. \\ \qquad \left. + \frac{K_1(1+\bar{h})}{\bar{h}} \bar{f} \right\} e^{-\Gamma_1(\bar{t}-\bar{t}')} d\bar{t}' \\ + \left(\frac{\bar{a}^2}{1-\bar{a}^2}\right)(k-1)\left(\frac{1}{\bar{a}^2} - \frac{1}{\bar{r}^2}\right) e^{-\Gamma_1 \bar{t}} - k \\ \qquad \text{for } \left(\bar{a} \le \bar{r} \le \frac{1+\bar{a}}{2}\right) \\ \frac{1}{2}\left[\frac{1}{\bar{a}^2} - \frac{4}{(1+\bar{a})^2}\right] \int_{0+}^{\bar{t}} \left\{ \frac{(1+\bar{h})}{\bar{E}\bar{h}} \frac{d\bar{f}}{d\bar{t}'} + \frac{K_2 2^m (1+\bar{h})^{2m+1} \bar{E}^{2m}}{\bar{h}\left[\left(\frac{1-\bar{a}^2}{\bar{a}^2}\right)(1+\bar{h})+2\bar{E}\bar{h}\right]^{2m}} \bar{f}^{2m+1} \right. \\ \qquad \left. + \frac{K_1(1+\bar{h})}{\bar{h}} \bar{f} \right\} e^{-\Gamma_1(\bar{t}-\bar{t}')} d\bar{t}' \\ + \frac{1}{2}\left[\frac{4}{(1+\bar{a})^2} - \frac{1}{\bar{r}^2}\right] \int_{0+}^{\bar{t}} \left\{ \frac{(1+\bar{h})}{\bar{E}\bar{h}} \frac{d\bar{f}}{d\bar{t}'} + \frac{K_2 2^m (1+\bar{h})^{2m+1} \bar{E}^{2m}}{\bar{h}\left[\left(\frac{1-\bar{a}^2}{\bar{a}^2}\right)(1+\bar{h})+2\bar{E}\bar{h}\right]^{2m}} \bar{f}^{2m+1} \right. \\ \qquad \left. + \frac{K_1(1+\bar{h})}{\bar{h}} \bar{f} \right\} e^{-\Gamma_2(\bar{t}-\bar{t}')} d\bar{t}' \\ + \left(\frac{\bar{a}^2}{1-\bar{a}^2}\right)(k-1)\left\{ \left[\frac{1}{\bar{a}^2} - \frac{1}{\bar{r}^2}\right] e^{-\Gamma_1 \bar{t}} + \left[\frac{4}{(1+\bar{a})^2} - \frac{1}{\bar{r}^2}\right] e^{-\Gamma_2 \bar{t}} \right\} - k \\ \qquad \text{for } \left(\frac{1+\bar{a}}{2} \le \bar{r} \le 1\right) \end{cases} \tag{94}$$

Note that $\bar{\sigma}_r$ is continuous at the junction of the regions, whereas the slope $\partial\bar{\sigma}_r/\partial\bar{r}$ is discontinuous. A similar approximate procedure applied to eq. (73.b) for σ_θ would yield a less useful result, since we see from the last term that σ_θ would be discontinuous at the junction of the rings.

Turning now to the evaluation of displacement , we first combine eqs. (80) and (81) to obtain the strain-rate in terms of f as

$$\frac{\partial\varepsilon_\theta}{\partial t} = \frac{b^2}{r^2}\left[\frac{3}{2E^c}(1+\frac{b}{h})\frac{\dot{f}}{2} + \tilde{C}^c_{Rs}(1+\frac{b}{h})\frac{f}{2} + \frac{\tilde{C}^c_{Ts}}{2^{2m+1}}(1+\frac{b}{h})^{2m} f^{2m+1}\right] \qquad (95)$$

After nondimensionalizing in accordance with eqs. (32) and (64) this relation becomes

$$\frac{\partial}{\partial\bar{t}}\bar{\varepsilon}_\theta = \frac{3}{4\bar{r}^2}\left\{\frac{(1+\bar{h})}{\bar{E}\bar{h}}\dot{\bar{f}} + \frac{K_2 2^m(1+\bar{h})^{2m+1}\bar{E}^{2m}}{\bar{h}\left[(\frac{1-\bar{a}^2}{\bar{a}^2})(1+\bar{h})+2\bar{E}\bar{h}\right]^{2m}}\bar{f}^{2m+1} + \frac{K_1(1+\bar{h})}{\bar{h}}\bar{f}\right\} \qquad (96)$$

Note that the tangential strain-rate and its derivatives are continuous throughout the whole fuel region. Finally, the nondimensional radial velocity is obtained from eq. (4) as

$$\bar{v} = \frac{\partial\bar{u}}{\partial\bar{t}} = \bar{r}\,\frac{\partial\bar{\varepsilon}_\theta}{\partial\bar{t}} \qquad (97)$$

where $\partial\bar{\varepsilon}_\theta/\partial\bar{t}$ has been given in eq. (96). The initial and asymptotic values $\bar{v}(\bar{r},0^+)$ and $\bar{v}(\bar{r},\infty)$ follow in the usual manner.

In the next section, some typical values are chosen for the nondimensional material parameter ratios and the Poisson coefficient, in order to study the nature of the stress and velocity distributions in the three parts of this section.

4. DISCUSSION OF RESULTS

In order to illustrate the solutions obtained in the previously discussed Cases I, II and III, we select for the nondimensional geometric and elastic constants the typical values $\bar{a} = 0.3$, $\bar{h} = 0.1$ and $\bar{E} = 1.0$ [see eqs. (37.a,b) and (38.a)]. Turning first to Case I (material properties independent of position - transient creep excluded), we shall select two sets of values for the nondimensional creep constants K_1 and K_2 in recognition of the large range of variation for these constants. Using the one-dimensional test conducted at a stress equal to the internal pressure p_o as the measure of material behavior, we shall designate as Case I.a the situation in which the steady thermal creep strain rates are of the same order in the fuel and cladding materials, while being of one order higher than the corresponding steady irradiation creep strain rates. Thus, referring to eq. (11.a), we set in Case I.a

$$\frac{F_{To}}{\lambda_T} = \frac{F^c_{To} p_o^{n-1}}{(\lambda^c_T)^n} = 10 \frac{F_{Ro}}{\lambda_R} = 10 \frac{F^c_{Ro}}{\lambda^c_R} \tag{98}$$

whereupon eqs. (38.b,c) yield the values $K_1 = 0.091$ and $K_2 = 0.909$. In addition, we consider the case in which the steady and irradiation creep strain rates are of the same order in the fuel material, and of one order higher than the corresponding creep rates in the cladding material. Designating this as Case I.b, we accordingly set

$$\frac{F_{To}}{\lambda_T} = \frac{F_{Ro}}{\lambda_R} = 10 \frac{F^c_{To} p_o^{n-1}}{(\lambda^c_T)^n} = 10 \frac{F^c_R}{\lambda^c_R} \tag{99}$$

and obtain $K_1 = K_2 = 0.05$.

The nondimensional stress and velocity solutions in the fuel for Case I.a are given by the solid curves in Figs. 3-6. For these and all subsequent curves $0.3 \leq \bar{r} \leq 1.0$ (from central hole to interface) and $0^+ \leq \bar{t} < \infty$ (from initial to asymptotic solutions), for $m = 0$ and 2 (linear and nonlinear thermal creep in cladding) and with $\nu = 0$ and 0.5 (non-contracting and incompressible materials). Case I.a is especially interesting in that for $m = 0$ we obtain $\alpha_I = \delta_I = 1$ as a consequence of $K_1 + K_2 = 1$ [see eqs. (35-36)], and thus eq. (41.a) gives a constant contact pressure $\bar{f} = 1$. Fig. 3 gives the radial and circumferential stress distributions plotted versus position $\bar{r}$ in the fuel; the value of ν had a minor effect on the stresses in this case and thus only the curves for $\nu = 0.5$ have been shown here. We see that $\bar{\sigma}_r$ is compressive and $\bar{\sigma}_\theta$ is tensile at all points. Also we find that the initial elastic and asymptotic stress solutions for $m = 0$ are identical, which is a direct consequence of the fact that $\bar{f} = 1$. Figures 4 and 5 present the radial and circumferential stresses respectively at the interfacial position $\bar{r} = 1$, but plotted now versus time $\bar{t}$. Note that for $m \neq 0$ $|\bar{\sigma}_r| = \bar{f}$ increases with time whereas $\bar{\sigma}_\theta$ decreases with time, and that the effect of ν is minor when compared to the effect of m. Finally, the radial velocity at the interface is plotted versus time in Fig. 6, and we see that the values of both ν and m have a very significant effect on these time decreasing solutions.

A corresponding set of stress and velocity solutions for Case I.b is given in Figs. 7-10. This case differs sharply from the previous case in that $\bar{f}$ always grows with time (see Fig. 8) and is never

constant. Furthermore, Figs. 7-9 show that the value of ν now has a very significant effect on the stress solutions. In fact, the Poisson ratio plays a very interesting role in Case I.b in that for $\nu = 0.1822$ the solutions are independent of the creep power m, and curves for this value of ν have also been shown. On comparing Cases I.a and I.b we find that the various solutions approach different asymptotic values, and the rate of approach is generally faster in Case I.a. We also note that although the initial stresses are identical in both cases the initial velocities are different since these are affected by the stress rate.

We turn next to Case II in which the material properties are independent of position while transient creep terms with equal retardation times are included, and for the sake of brevity we will consider only one set of material creep constants. We again use the one-dimensional test at p_o as the measure of material behavior, but now also specify that the creep terms in eq. (11.a) be compared at $t = 2\tilde{\tau}$ since the transient terms have essentially reached their asymptotic values at this time. We shall take Case II as an extension of Case I.a, and thus we assume that under the above conditions all of the thermal creep strain components in the fuel and cladding materials are of the same order as the elastic strain, whereas all of the irradiation creep strain components are of one order lower. Accordingly, referring to eq. (11.a) we set

$$\frac{2\tilde{\tau}F_{To}}{\lambda_T} = \frac{2\tilde{\tau}F^c_{To}p_o^{n-1}}{(\lambda^c_T)^n} = \frac{1}{\mu_T} = \frac{p_o^{n-1}}{(\mu^c_T)^n} = \frac{1}{E}$$

$$= \frac{20\tilde{\tau}F_{Ro}}{\lambda_R} = \frac{20\tilde{\tau}F^c_{Ro}}{\lambda^c_R} = \frac{10}{\mu_R} = \frac{10}{\mu^c_R} \tag{100}$$

whereupon eqs. (38.b,c) and (63.a-d) yield $K_1 = 0.091$, $K_2 = 0.909$, $K_3 = 1.818$, $K_4 = 0.182$, $K_5 = 2.000$ and $K_6 = 1.818$.

In Case II the curves for $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ vs. $\bar{r}$ at $\bar{t} = 0$ and ∞ are shown on Fig. 3 (identical with Case I.a), the curves for $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ vs. $\bar{t}$ at $\bar{r} = 1.0$ are given as dashed lines on Figs. 4,5, and lastly the graphs of $\bar{v}$ at $\bar{r} = 1.0$ are presented separately in Fig. 11. Figures 3,4 and 5 clearly demonstrate that the asymptotic values of $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ at the interface are identical in Cases I.a and II, which is as expected since Case II reduces to I.a as the transient effects die out. We also note in Figures 4 and 5 that the rate of approach to the asymptotic stress values is generally slower in Case II than in Case I.a, although in the special case $m = 0$ we again have $\bar{\Gamma} = 1$. On comparing the velocity solutions for these two cases (Figures 6 and 11), we see that, while $\bar{v}(1,\bar{t})$ is constant for $m = 0$ and slightly decreasing for $m = 2$ in Case I.a, all velocity solutions decrease through a much greater range in Case II. This is a direct consequence of the fact that the two cases have different initial velocities (Case II much greater) but equal asymptotic velocities, as discussed previously in section 3.2.

We finally turn to Case III, in which the transient creep is ignored while the creep properties are allowed to vary with the radial position. You will recall that we simplified the analysis in section 3.3 by assuming incompressibility $(\nu = 0.5)$ and by employing a multi-ring approximation. In a fuel rod the temperature and flux and accordingly the creep parameters Γ_T and Γ_R will decrease in the radial direction. For the purpose of illustration we shall assume

that $\Gamma_T = \Gamma_R = \Gamma$ where Γ is a parabolic function [i.e., M = 2 in eqs. (23)] with $d\Gamma/d\bar{r} = 0$ at $\bar{r} = .3$, with $\Gamma(.3)/\Gamma(1) = 2.6$, and with the mean $(1/.7)\int_{.3}^{1} \Gamma(r')dr' = 1$ as previously required. We thereby obtain

$$\Gamma(\bar{r}) = 1.116 + .948\bar{r} - 1.580\bar{r}^2 \tag{101}$$

whereby $\Gamma(.3) = 1.258$ and $\Gamma(1) = .484$, which is a change of moderate magnitude. We consider that the average material behavior is the same as in Case I.a [see eqs. (98)], and employ the two ring approximation for the radial dependence, with Γ_1 the average value in the inner ring $(0.3 \leq \bar{r} \leq 0.65)$ and Γ_2 the average value in the outer ring $(0.65 \leq \bar{r} \leq 1.0)$. The complete set of nondimensional creep constants for use in eqs. (93) is then given as $K_1 = 0.091$, $K_2 = 0.909$, $\Gamma_1 = 1.193$, $\Gamma_2 = 0.806$.

For Case III we evaluated $\bar{\sigma}_r$ by means of the simple estimate given by Eq. (4) (which uses the step approximation to $\Delta(r)$ throughout) and $\bar{v}$ by means of eq. (97); the curves for $\bar{\sigma}_r(\bar{r},\infty)$, $\bar{\sigma}_r(1,\bar{t})$ and $\bar{v}(1,\bar{t})$ (for $m = 0,2$) are shown as dash-dot lines on Figures 3,4 and 6 respectively. Note that as expected the slope of the radial stress is discontinuous in Figure 3 at the point $\bar{r} = 0.65$. On comparing Case III with Case I.a we see from Figure 4 that the asymptotic value of $\bar{f}$ is greater in Case III, and thus for $m = 0$ the contact pressure in Case III is no longer equal to unity. Finally, on examining Figure 6 we see that neither the initial nor the asymptotic velocities are equal for the two cases, and hence for $m = 0$ the velocity at the interface is again no longer constant for Case III. We must point out

that these curves are solely for the purpose of illustration, and that it would generally be necessary to use more than two rings to obtain good estimates for the contact pressure and the integration function $g_2(t)$. Furthermore, as we have pointed out, one could not only obtain a better estimate for the radial stress but could also obtain a reasonable continuous estimate for the tangential stress by using the actual radial distribution for the creep properties in eqs. (63).

As previously stated, our major goal here was to obtain some analytical solutions for various special cases, which could then be compared with numerical results from existing computer codes. In order to obtain such solutions for this very complex problem, we found it necessary to make a number of simplifying assumptions. As one attempts to relax these assumptions, the complexity of the problem increases only slightly in some cases while considerably more in some other cases. For example, the solutions presented may be easily extended to include time dependence in the inner radius and pressure, and an external pressure may also be easily included. On the other hand, the inclusion of swelling and thermal expansion would result in a substantial increase in complexity since nonlinearity precludes the use of simple superposition. Finally, the relaxation of some other assumptions would result in problems which are orders of magnitude more difficult. In this category we have the inclusion of nonlinearity in the thermal creep of the fuel as well as of the cladding, since the compatibility relation in the stress function is then a nonlinear partial differential equation.

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REFERENCES

1. Zudan, Z., Chen, T.C., and Steigelmann, W.H., Thermal Stress Technique in the Nuclear Industry, Franklin Institute Research Laboratories, Elsevier, 1965.

2. Morland, L.W., and Lee, E.H., "Stress Analysis for Linear Visco-elastic Materials with Temperature Variation," Transactions of the Society of Rheology IV, pp. 233-263, 1960.

3. Cozzarelli, F.A., and Huang, S., "On the Analogy between Therm-ally and Irradiation Induced Creep," Nuclear Engineering and Design 42 (1977), pp. 409-421.

4. Rashid, Y.R., "Mathematical Modeling and Analysis of Fuel Rods," Nuclear Engineering and Design 29 (1974), pp. 22-32.

5. Olander, D.R., Fundamental Aspects of Nuclear Reactor Fuel Elements, Technical Information Center, Energy Research and Development Administration, 1976.

6. Gilbert, E.R., and Straalsund, J.L., "A Relationship for Non-Conservative Volume Creep under Different States of Stress," Nuclear Engineering and Design 12, pp. 421-424, 1970.

7. Gittus, J.H., "Multiaxial Primary Irradiation Creep Equations for Reactors Structural Material ," Properties of Reactor Structural Alloys after Neutron or Particle Irradiation, ASTM Publication (STP 570), 1975.

8. Courtine, D., Cozzarelli, F.A. and Shaw, R.P., "Effects of Time Dependent Compressibility on Nonlinear Viscoelastic Wave Propaga-tion," International Journal of Nonlinear Mechanics Vol. 11, pp. 365-383 (1976).

9. Lee, E.H., Radok, J.R.M. and Woodward, W.B., "Stress Analysis for Linear Viscoelastic Materials," Transaction of the Society of Rheology III, pp. 41-59, 1959.

10. Rogers, T.G. and Lee, E.H., "The Cylinder Problem in Viscoelastic Stress Analysis," Quarterly Applied Mathematics 22, (1964), pp. 117-131.

11. Huang, N.C., Lee, E.H. and Rogers, T.G., "On the Influence of Viscoelastic Compressibility in Stress Analysis," Proceedings of 4th International Congress of Rheology, New York, Wiley and Sons.

12. Ting, T.C.T., "Remarks on Linear Viscoelastic Stress Analysis in Cylinder Problems," Proceedings of 9th Midwestern Mechanics Conference 1965, p.263.

13. Ting, E.C., "Stress Analysis for Linear Viscoelastic Cylinders," AIAA Vol. 8, No. 1, Jan. 1970, p. 18.

14. Wordesworth, J., "LAMBUS 1-A Digital Computer Code for the Design, In-pile Performance Prediction and Post-irradiation Analysis of Arbitrary Fuel Rods - Part I: Theory and Modeling," Nuclear Engineering and Design 31 (1974), pp. 301-336.

15. Greenbaum, G.A. and Rubinstein, M.F., "Creep Analysis of Axisymmetric Bodies Using Finite Element," Nuclear Engineering and Design 7 (1968), pp. 379-397.

16. Sutherland, W.H., "AXICRP - Finite Element Computer Code for Creep Analysis of Plane Stress, Plane Strain and Axisymmetric Bodies," Nuclear Engineering and Design 11 (1970), pp. 269-285.

17. Ma, B.M., "Irradiation Swelling, Creep and Thermal Stress Analysis of LWR Fuel Element Computer Code - ISUNE 2," Nuclear Engineering and Design 34 (1975), pp. 361-378.

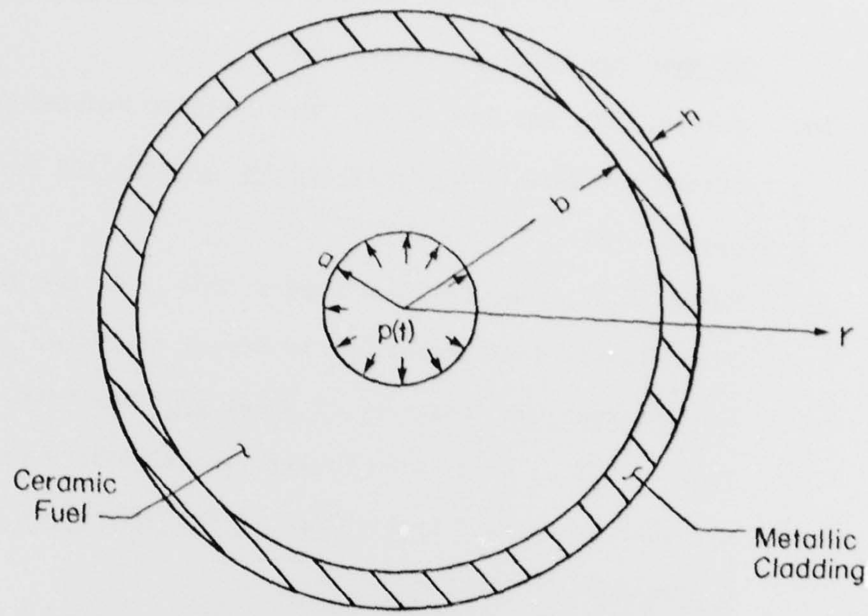
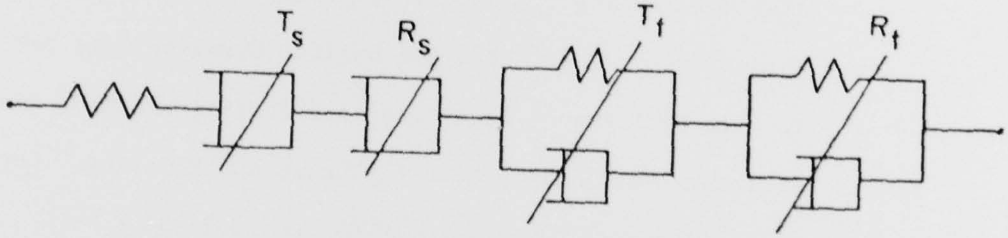


Figure 1 - Geometry of the Fuel Rod



Fuel:	E	λ_T	λ_R	μ_T, τ_T	μ_R, τ_R
		$n_T = 1$	$n_R = 1$	$q_T = 1$	$q_R = 1$
Cladding:	E^c	λ_T^c	λ_R^c	μ_T^c, τ_T^c	μ_R^c, τ_R^c
		$n_T^c = n$	$n_R^c = 1$	$q_T^c = n$	$q_R^c = 1$

Figure 2 - Mechanical Model For Both Materials

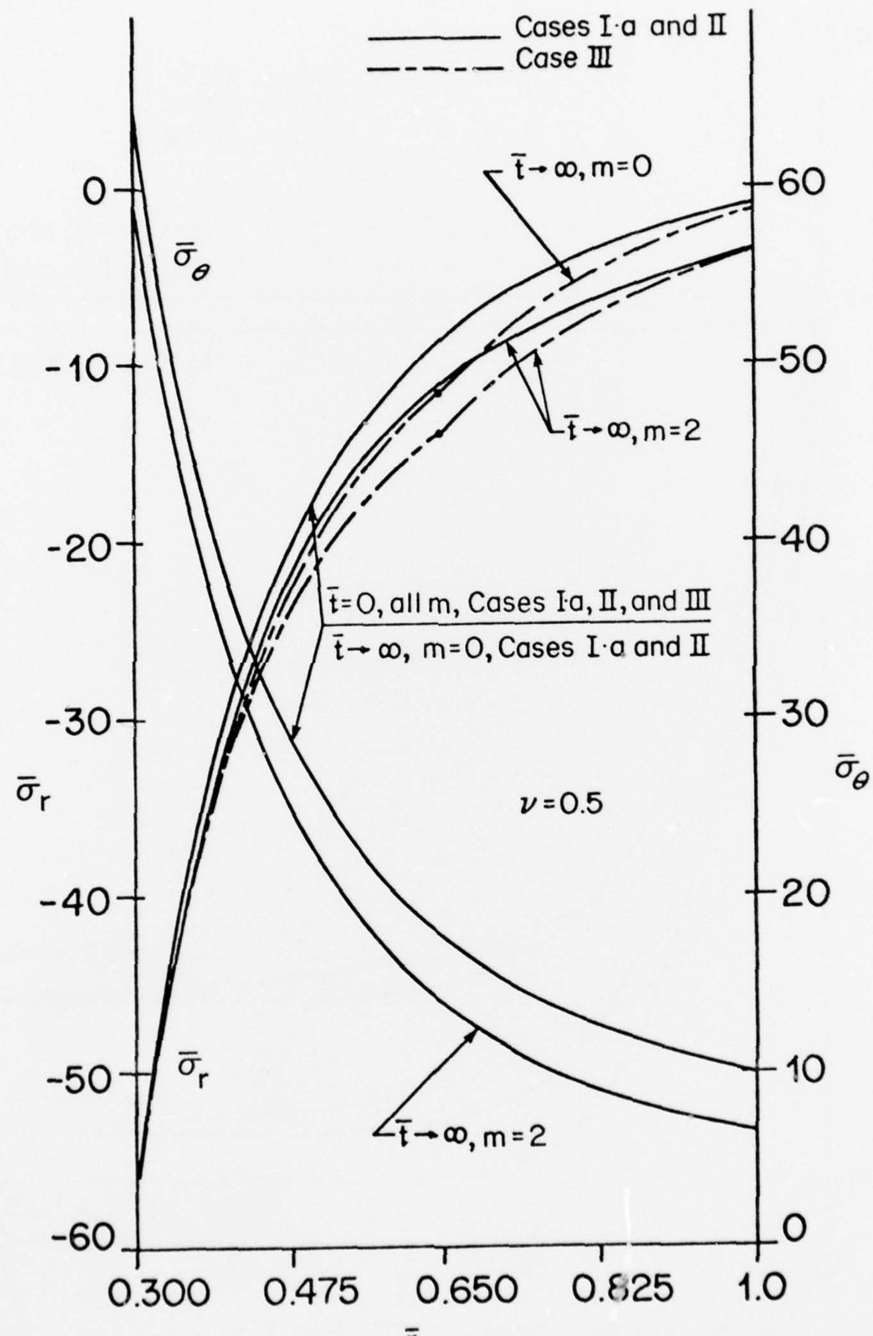


Figure 3 - Stresses ($\bar{\sigma}_r$ and $\bar{\sigma}_\theta$) vs. $\bar{r}$ - Cases I·a, II, and III

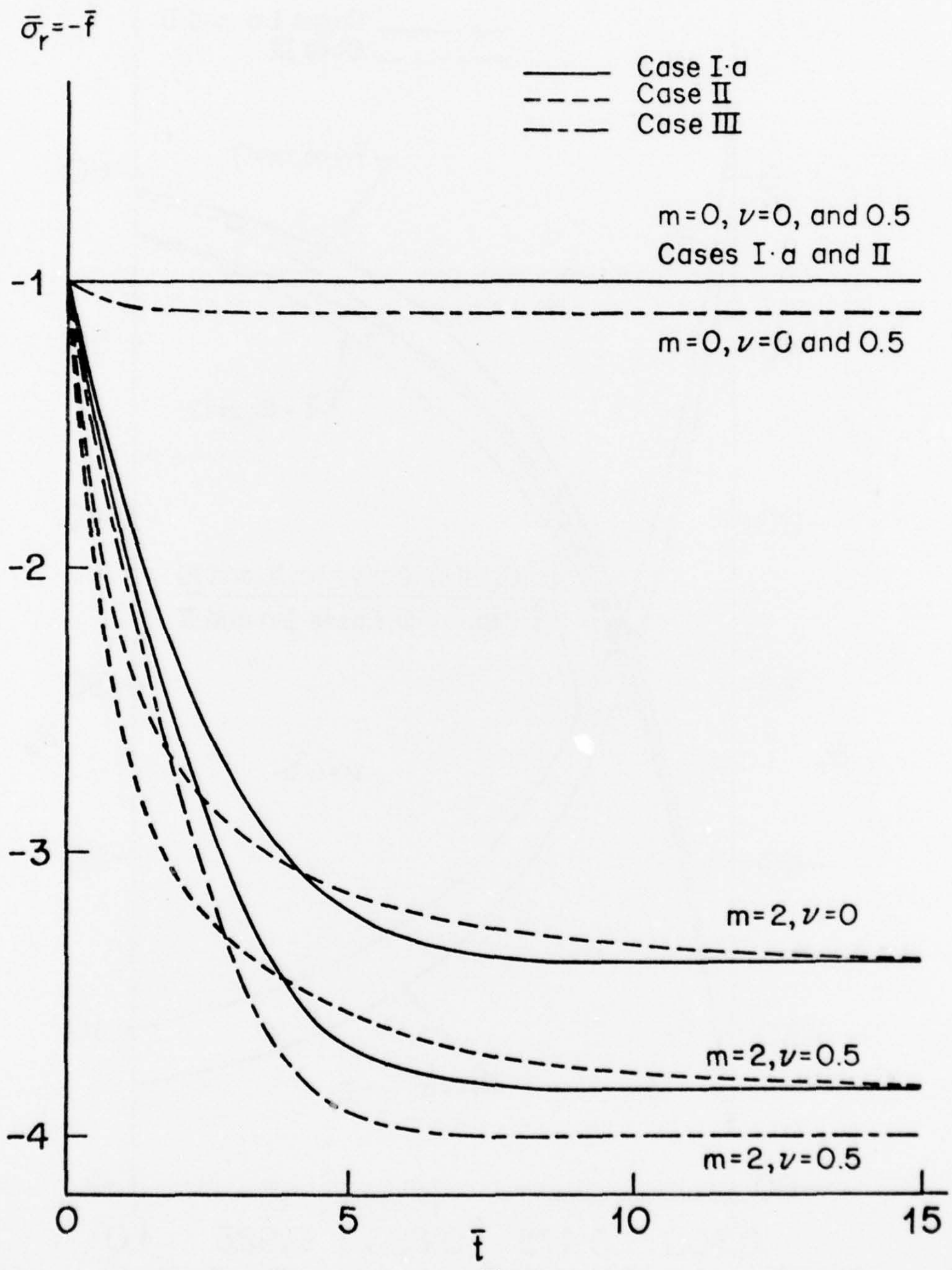


Figure 4 - Radial Stress $\bar{\sigma}_r$ vs. $\bar{t}$ at $\bar{r}$=1.0 - Cases I·a, II and III

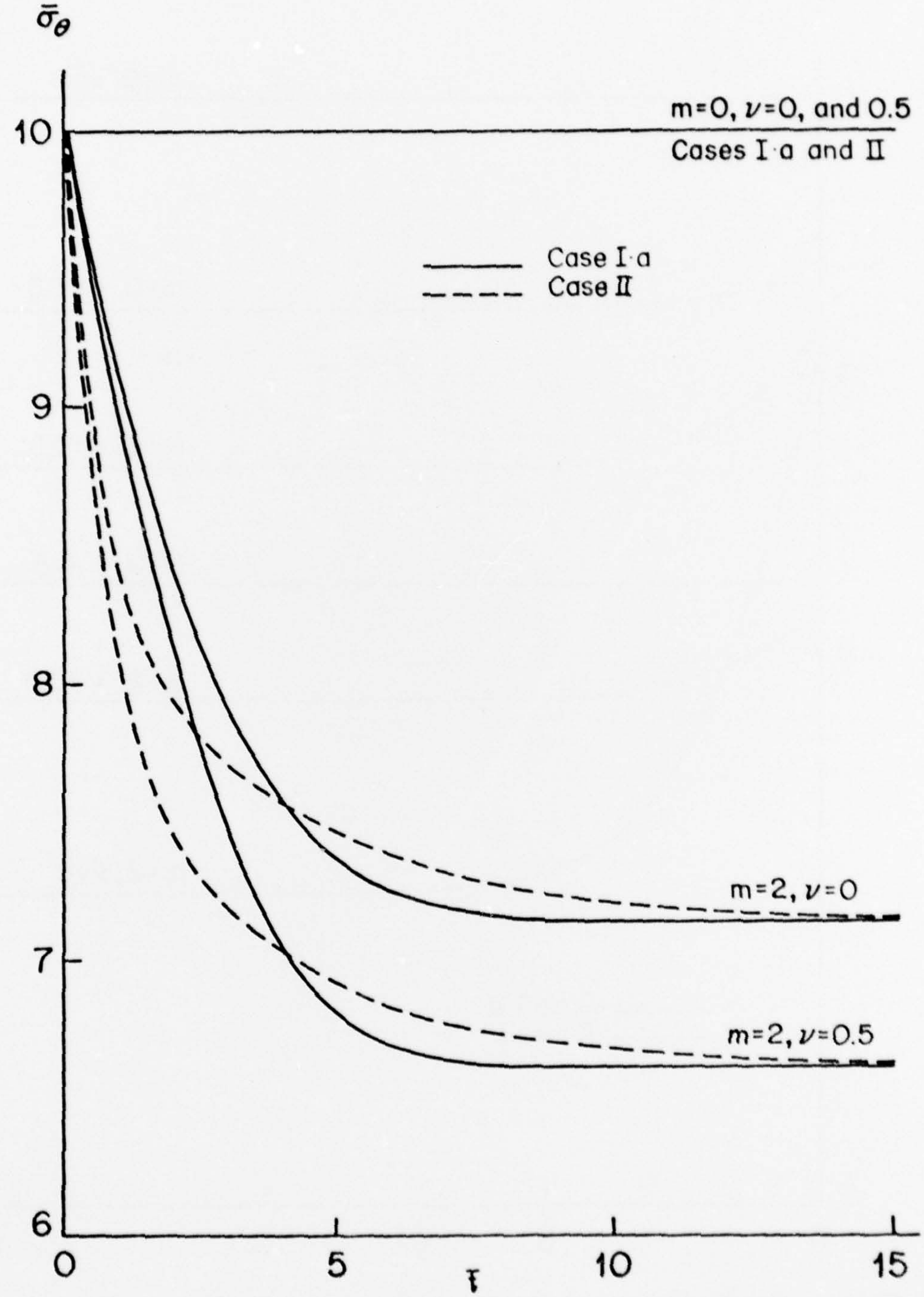


Figure 5 - Tangential Stress $\bar{\sigma}_\theta$ vs. $\bar{t}$ at $\bar{r}$=1.0 - Cases I·a and II

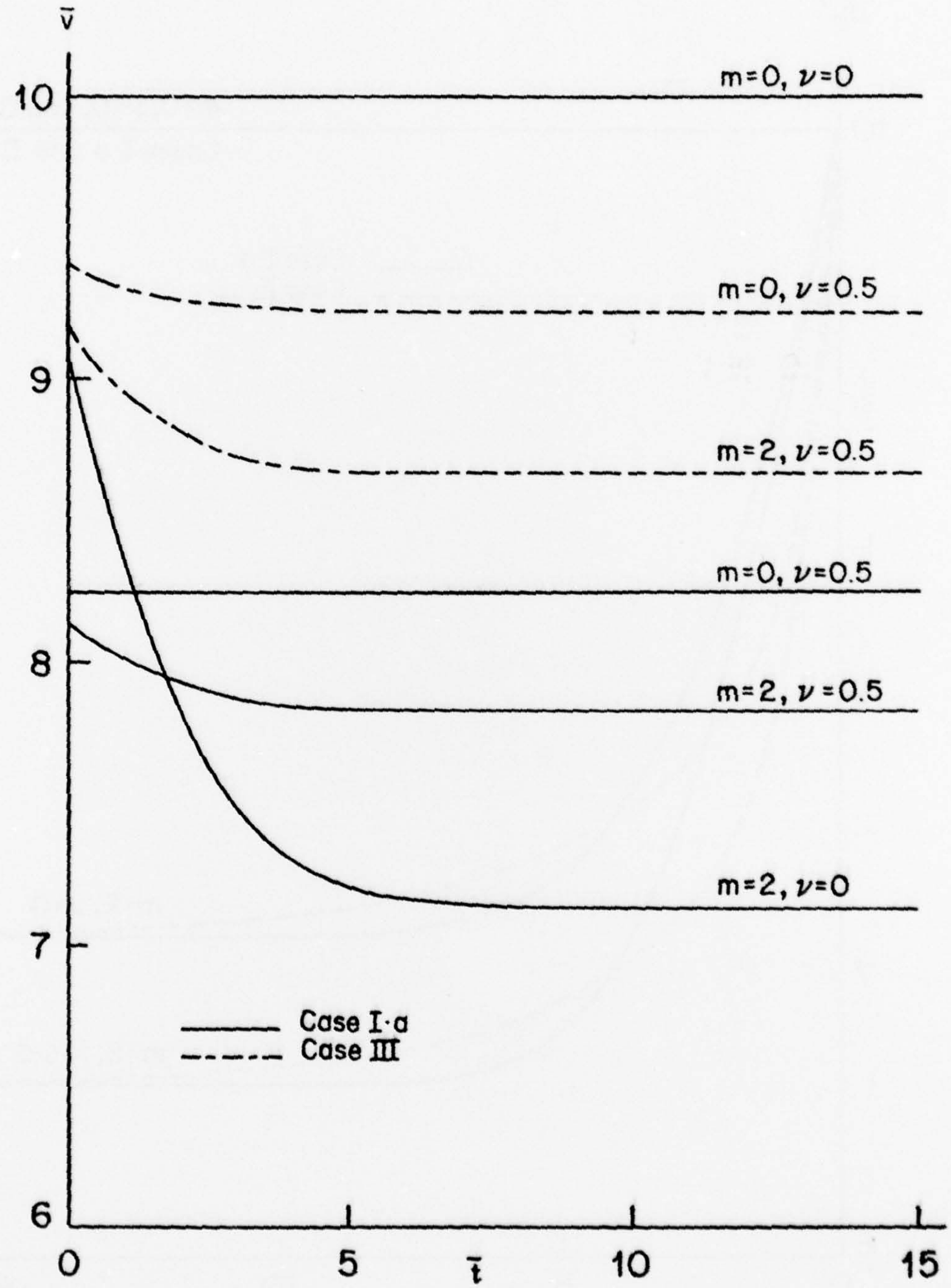


Figure 6 – Radial Velocity $\bar{v}$ vs. $\bar{t}$ at $\bar{r}$=1.0 – Cases I·a and III

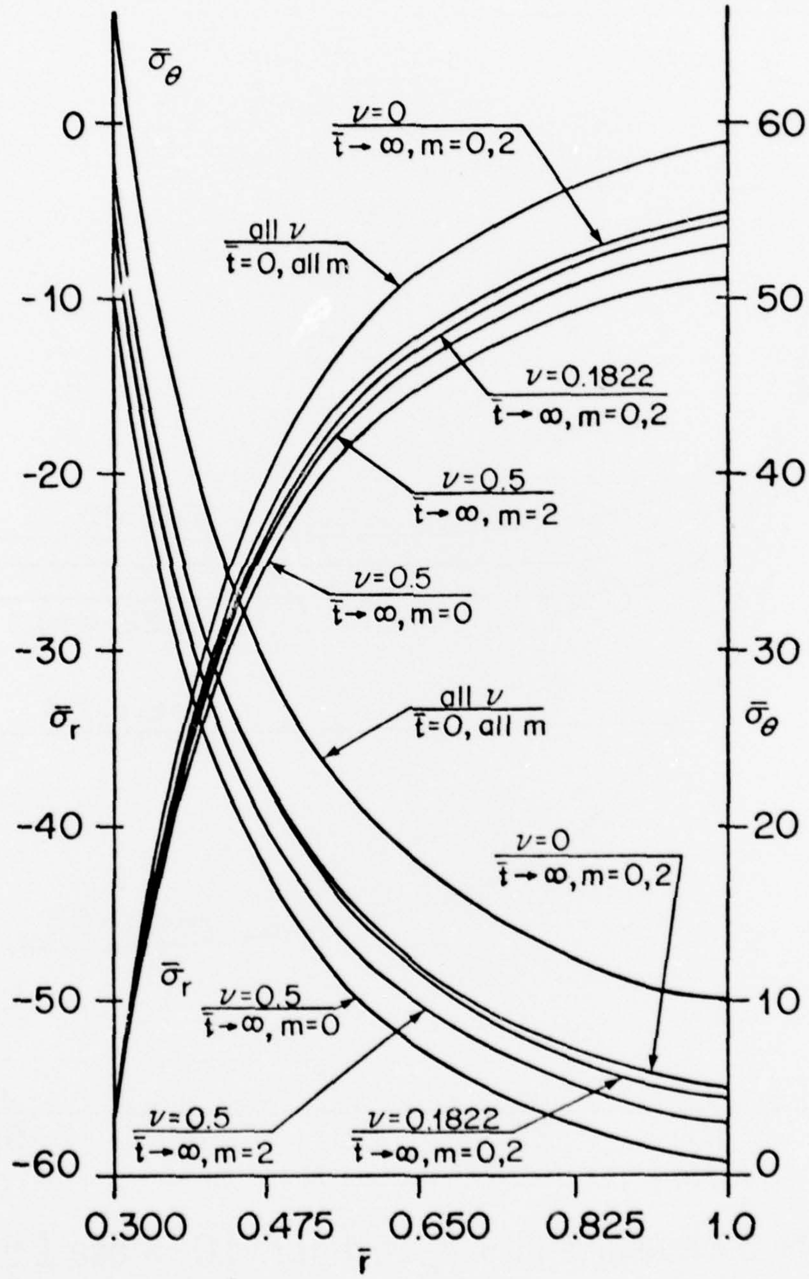


Figure 7 - Stresses ($\bar{\sigma}_r$ and $\bar{\sigma}_\theta$) vs. $\bar{r}$ – Case I·b

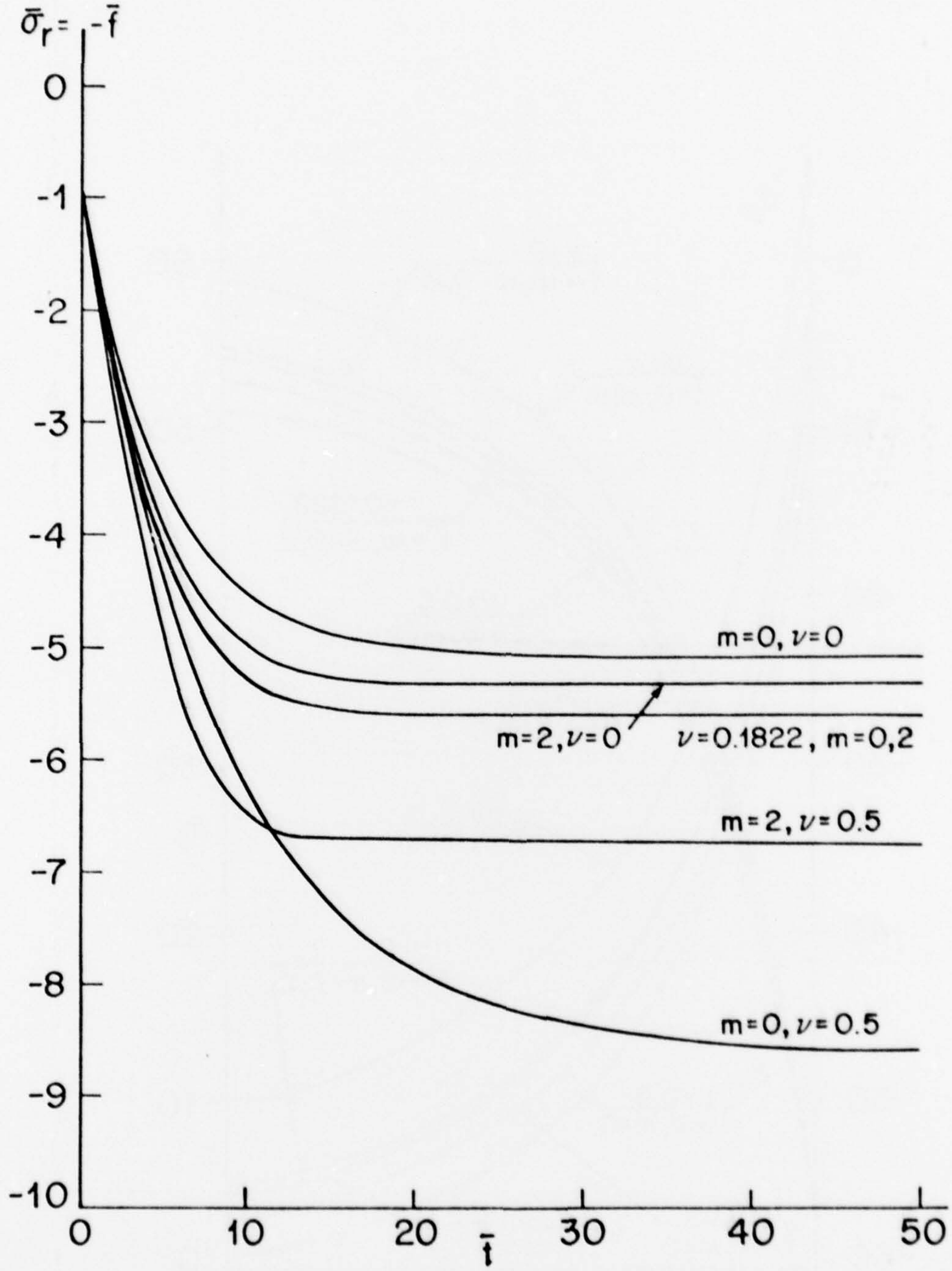


Figure 8 - Radial Stress $\bar{\sigma}_r$ vs. $\bar{t}$ at $\bar{r}=1.0$ - Case I·b

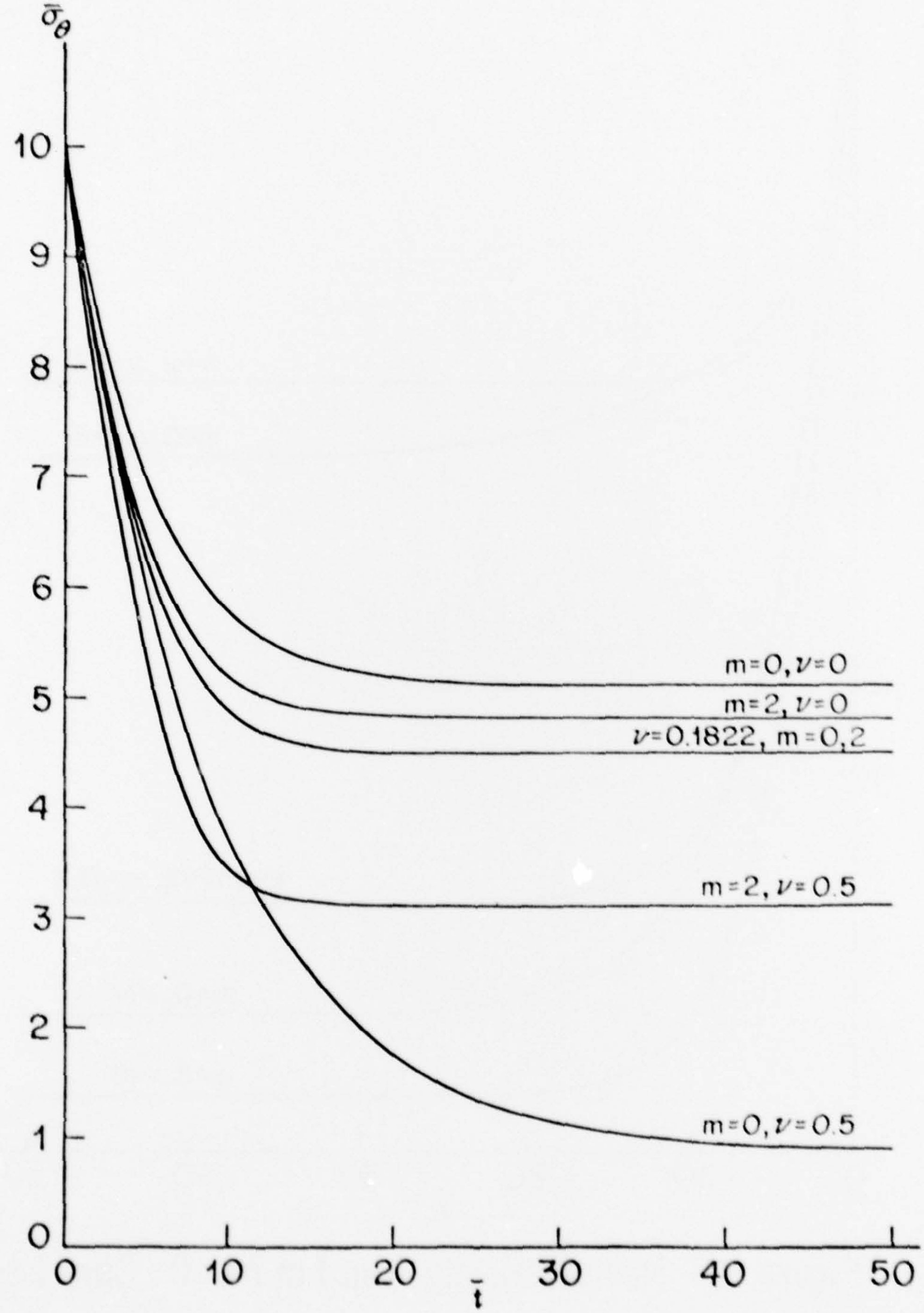


Figure 9 - Tangential Stress $\bar{\sigma}_\theta$ vs. $\bar{t}$ at $\bar{r}$ =1.0 - Case I·b

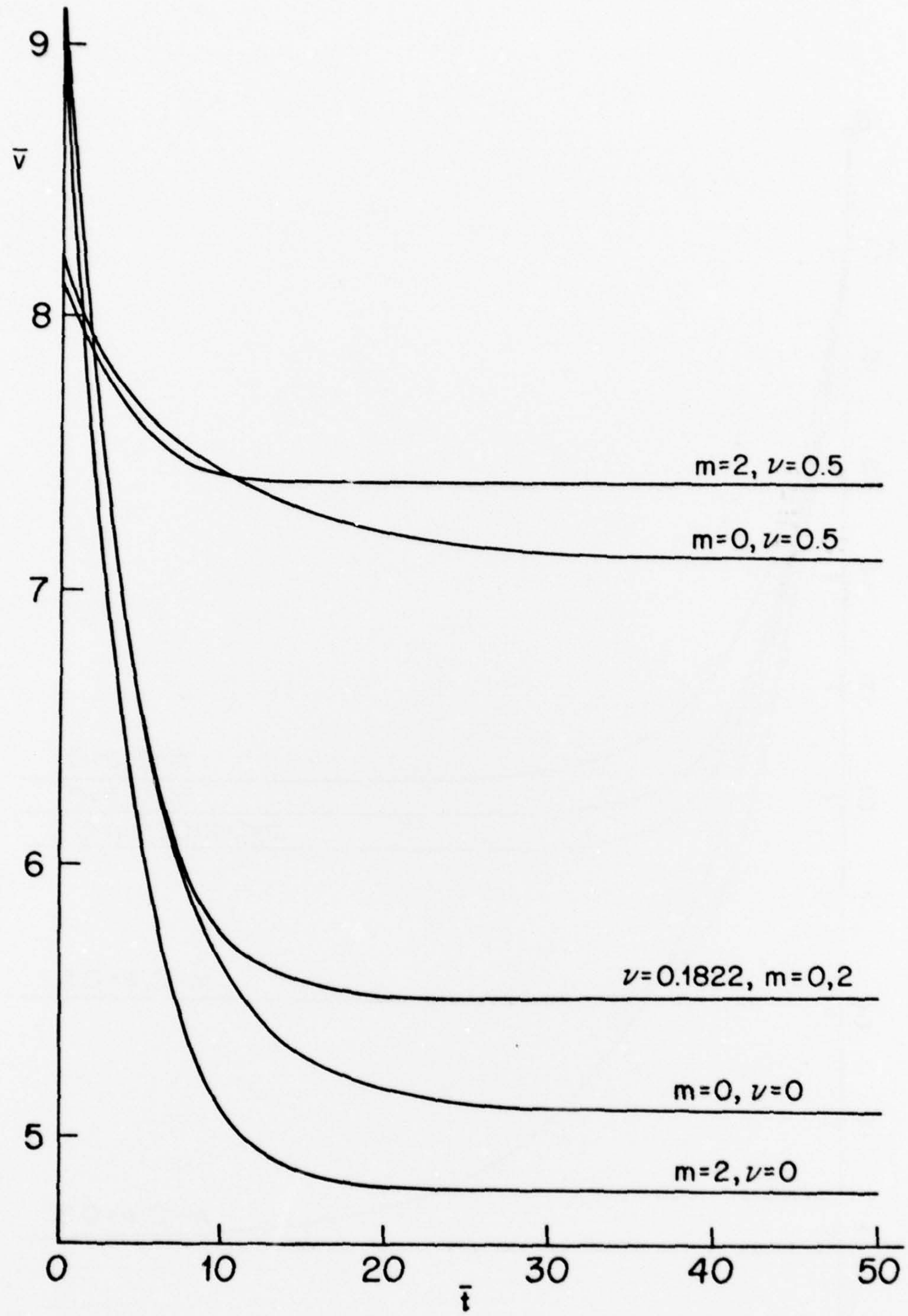


Figure 10 - Radial Velocity $\bar{v}$ vs. $\bar{t}$ at $\bar{r}$ = 1.0 - Care I·b

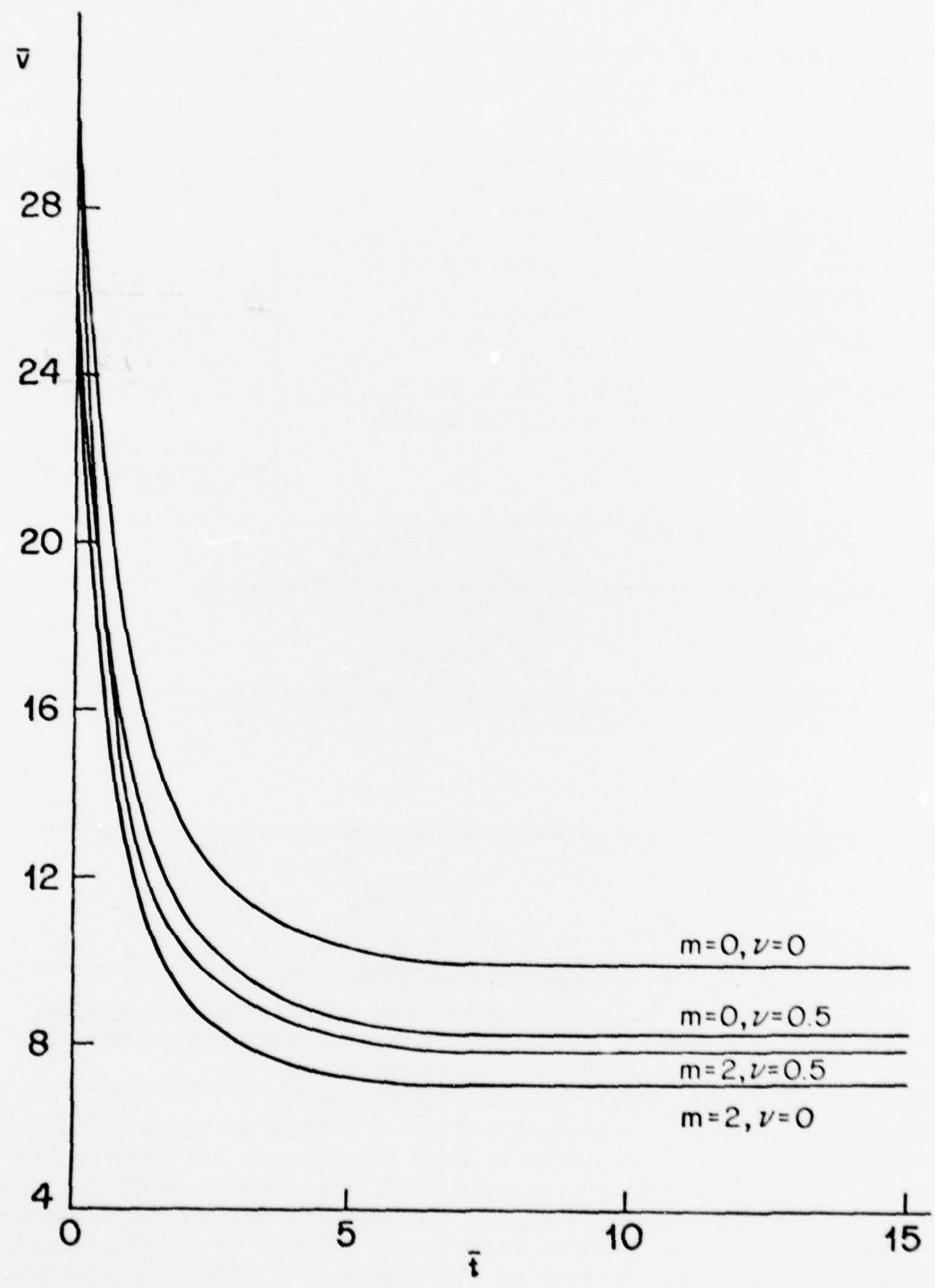


Figure 11 - Radial Velocity $\bar{v}$ vs. $\bar{t}$ at $\bar{r}$ = 1.0 - Case II

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stresses and velocities, cladded fuel rod, irradiation and thermal creep, plane strain, compressible nonlinear material, temperature and flux dependent properties, steady and transient creep.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Stresses and velocities are analyzed for a hollow cylinder of fuel encased in metallic cladding and subjected to high temperature and high neutron flux fields. The material is represented by a compressible nonlinear thermoirradiation viscoelastic model. A stress function for axisymmetric plane strain is introduced, and the problem essentially reduces to solving a nonlinear ordinary differential equation in the interfacial contact pressure. Some analytic results are obtained for the case of material properties independent of position. For the case of temperature, flux and thus material properties dependent on position, an approximate formulation is employed whereby the cylinder is divided into discrete rings with constant properties.